The Pseudovector Mirage

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When God saw men building the tower of Babel, he confounded their speech to take them down a peg. It is just possible that axial vectors were inspired by a divine fear of physicists being too successful. Indeed, one has to go a long way to find a concept in physics that generates more confusion in a shorter time than the idea of the **pseudovector**, alias **axial vector**. The prefix pseudo rightly conjures up the notion of something which is *not* a vector but resembles it in some ways. In this case, it is supposed to be an object resembling a vector in that it has a magnitude, direction, and orientation (a positive or negative sense) but does not transform as a vector under some symmetry operations. "Pseudovector" is a perfectly valid nomenclature for the said object but confusion sets in when the object, admittedly not a vector, is actually *represented* by a vector. Once this is done, the pseudovector becomes a will-o'-the-wisp, eluding understanding each time one attempts to focus on it.

How can something which by definition is not a vector be represented by a vector? Who in his right mind would do such a thing? Unfortunately, almost everyone, if 99% of all physics books are to be believed. Many non-scalar quantities that depend in a fundamental way on the product of *two* vectors are invariably said to be *vectors* and are defined via the Gibbs **cross-product**:

$$B = a \times b, \tag{1}$$

where a and b are the two vectors and B is the vector representation of the object. Angular momentum is a typical example, where a and b are the position and linear momentum vectors, respectively. Obviously, this enterprise is doomed to failure because quantities such as angular momentum simply do not transform like a vector and should not be represented by one.

It is the Gibbs cross-product itself that is the culprit, for it implies that a vector can result from the *product* of two vectors. It cannot. There is a contradiction that is immediately apparent upon performing a transformation upon the vectors. Consider a symmetry transformation, inversion in the origin. Vectors change their orientation under such a transformation: $a \to -a$, $b \to -b$ and $B \to -B$ so the equation above transforms to

$$-B = (-a) \times (-b) = a \times b = B, \tag{2}$$

which can only be true if $B = 0$, a nonsensical result that ought to set alarm bells jangling.

The proper way out of this cul-de-sac is to realize that the cross-product is in fact *not* a bona fide product but only a *recipe* for obtaining a vector in 3D that is perpendicular to two given vectors. As such it is *not permissible* to use it to define some object that depends on a product of two vectors. Instead, it is necessary to use the **exterior product** of the two vectors, since this is proportional to each of the vectors and *is* a proper product:

$$B = a \wedge b, \tag{3}$$

where B is now a **bivector**, not a vector. This equation can be used to *define* how B transforms. Again, under inversion

$$B = a \wedge b \rightarrow (-a) \wedge (-b) = a \wedge b = B, \tag{4}$$

leaving B unaltered. The cross product is still a useful computational tool but it must be defined in terms of the bivector:

$$B = ia \times b, \tag{5}$$

where $i = e_1 \wedge e_2 \wedge e_3$ is the pseudoscalar (in 3D a **trivector**), the exterior product of the three independent unit vectors spanning the space. In this way, angular momentum can be *associated* with a vector, even though it is not *represented* by one! The cross-product is **dual** to the bivector.

Thus, no inconsistency arises when quantities like angular momentum are regarded as bivectors, which are segments of *area* having magnitude, direction and orientation instead of segments of a *line* with magnitude, direction and orientation. In 3D, any bivector can be expressed as a linear combination of three independent unit bivectors just as a vector can be expressed as a linear combination of three independent unit vectors. The meaning of the term pseudovector is now clear: a pseudovector is "not a vector" because it is an area segment, not a line segment; it nevertheless shares certain properties with a vector: magnitude, direction, orientation and (in 3D) has three components. Still, the term bivector is preferable, since it is better to describe something by what it is rather than by what it is not!

Unfortunately, the historical and popular way of "working around" the obvious nonsense (Eqn. 2) implied by the definition in terms of the cross-product (Eqn. 1) is to invent two different types of vectors: **polar** vectors (ordinary vectors) and **axial** vectors. The latter are just like the former except that they remain unchanged under inversion in the origin. All vectors generated by the cross-product are included in this category. This smoke-and-mirrors tactic is the source of much confusion and is the direct consequence of insisting on encoding things like angular momentum in terms of vectors only, even when, as shown above, this is impossible.

The mental contortions required to maintain the illusion of the veracity of mutually contradictory things are truly amazing. One author has emphasized that an axial vector is not really a vector (correct) and that therefore the word "axial" is not an adjective (confabulation)! Is this a very helpful approach to a student meeting all this for the first time? Is it sound pedagogy to warn the reader that he is entering a linguistic mine-field because in discussing this topic, the normal use of english is just plain inadequate? Another author claims that the unit orthogonal basis vectors e_1, e_2, and e_3 in terms of which all vectors in 3-space can be expressed are in fact axial vectors because each of them can be expressed as a cross-product of the other two: $e_i \times e_j = e_k$. Logically, this implies that *all* vectors are axial vectors! All of this madness stems from the need to bear in mind whence each vector comes: a polar vector simply encodes some vector quantity like velocity, whereas an axial vector comes from a cross-product. But a vector is a vector is a vector and does not come equipped with any special label telling you where it came from. If presented with a room full of vectors, there is no way that you can tell which of these are axial and which are polar. Trying to understand axial vectors is akin to playing the old shell game—now you see it, now you don't.

Confusion about the pseudovector can successfully be banished by 1) changing its name to "bivector" and 2) representing it by a bivector. Step 1 is not just a palliative. It is a suggestion to call a spade a spade, always healthy pedagogy. Step 2 is only common sense. This means that a physical quantity such as angular momentum must be identified with $r \wedge p$ and not with $r \times p$ if it is to exhibit the correct transformation properties. This is not just a matter of nomenclature but is an *essential* step if inconsistencies and confusion are to be avoided. One must at least make an effort to use mathematical objects appropriate to the task at hand.

The effort needed may, however, be Herculean. In dispelling the fog surrounding the pseudovector, we have seen that it is the Gibbs cross-product that is to blame. That and the unwillingness to use mathematical objects such as the bivector that arise naturally from bona fide vector products. But the Gibbs cross-product is ubiquitous in physics, appearing not only in mechanics but also in other areas such as electromagnetism. In most cases, it should be replaced by the appropriate bivector. Thus the criticism vented here is not as esoteric as it might appear but rather applies to the whole of elementary physics, even at school level. Nothing less than a complete revamping of the basic formulations at all levels will do! This is truly a huge undertaking but it is worth it, for the gain in understanding is immense and the waste of time and energy chasing illusions could be vastly reduced.