Chapter 1

Jacobian Derivation for EKF and Gyro

As the system function used in our model is only affected by the state, it can be expressed as $\hat{\boldsymbol{x}}_k = \boldsymbol{A}\hat{\boldsymbol{x}}_{k-1} + \boldsymbol{w}_k$ where $\boldsymbol{A}\hat{\boldsymbol{x}}_{k-1} = \boldsymbol{f}_x$. Using the state transition matrix and state matrix of Chapter **??**, the full set of equations contained by $\boldsymbol{f}_x$ can be shown to be:

$$\begin{bmatrix} \boldsymbol{p}_k \\ \boldsymbol{q}_k \\ \boldsymbol{r}_k \\ \boldsymbol{\phi}_k \\ \boldsymbol{\theta}_k \\ \boldsymbol{\psi}_k \\ \dot{\boldsymbol{\phi}}_k \\ \dot{\boldsymbol{\theta}}_k \\ \dot{\boldsymbol{\psi}}_k \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{\theta}}_{k-1} - \dot{\boldsymbol{\psi}}_{k-1} sin\boldsymbol{\theta}_{k-1} \\ \dot{\boldsymbol{\theta}}_{k-1} cos\theta_{k-1} + \dot{\boldsymbol{\psi}}_{k-1} cos\boldsymbol{\theta}_{k-1} sin\boldsymbol{\phi}_{k-1} \\ -\dot{\boldsymbol{\theta}}_{k-1} sin\boldsymbol{\theta}_{k-1} + \dot{\boldsymbol{\psi}}_{k-1} cos\boldsymbol{\theta}_{k-1} \cos\boldsymbol{\phi}_{k-1} \\ \boldsymbol{\phi}_{k-1} + \dot{\boldsymbol{\phi}}_{k-1} dt \\ \boldsymbol{\theta}_{k-1} + \dot{\boldsymbol{\theta}}_{k-1} dt \\ \boldsymbol{\psi}_{k-1} + \dot{\boldsymbol{\psi}}_{k-1} dt \\ \boldsymbol{p}_{k-1} + \boldsymbol{q}_{k-1} sin\boldsymbol{\phi}_{k-1} tan\boldsymbol{\theta}_{k-1} + \boldsymbol{r}_{k-1} cos\boldsymbol{\phi}_{k-1} tan\boldsymbol{\theta}_{k-1} \\ \boldsymbol{q}_{k-1} cos\boldsymbol{\phi}_{k-1} - \boldsymbol{r}_{k-1} sin\boldsymbol{\phi}_{k-1} \\ \boldsymbol{q}_{\boldsymbol{k-1}} \frac{sin\boldsymbol{\phi}_{k-1}}{cos\boldsymbol{\theta}_{k-1}} + \boldsymbol{r}_{k-1} \frac{cos\boldsymbol{\phi}_{k-1}}{cos\boldsymbol{\theta}_{k-1}} \end{bmatrix} + \boldsymbol{w}_{k-1} \tag{1.1}$$

As in Equation **??**, the Jacobian $\nabla \boldsymbol{f}$ can be found using:

$$\frac{\delta \boldsymbol{f}_{[i]}}{\delta \boldsymbol{x}_{[j]}}|_{x=x_{k-1}} = \begin{bmatrix} \frac{\delta \boldsymbol{f}_1}{\delta x_1} & \cdots & \frac{\delta \boldsymbol{f}_1}{\delta x_n} \\ \vdots & \ddots & \vdots \\ \frac{\delta \boldsymbol{f}_n}{\delta x_1} & \cdots & \frac{\delta \boldsymbol{f}_n}{\delta x_n} \end{bmatrix}_{x=x_{k-1}} \tag{1.2}$$

and work through all the differentials to produce:

$$\nabla \boldsymbol{f} = \begin{bmatrix} 0 & 0 & 0 & \frac{\delta f_1}{\delta x_4} & \frac{\delta f_1}{\delta x_5} & \frac{\delta f_1}{\delta x_6} & \frac{\delta f_1}{\delta x_7} & \frac{\delta f_1}{\delta x_8} & \frac{\delta f_1}{\delta x_9} \\ 0 & 0 & 0 & \frac{\delta f_2}{\delta x_4} & \frac{\delta f_2}{\delta x_5} & \frac{\delta f_2}{\delta x_6} & \frac{\delta f_2}{\delta x_7} & \frac{\delta f_2}{\delta x_8} & \frac{\delta f_2}{\delta x_9} \\ 0 & 0 & 0 & \frac{\delta f_3}{\delta x_4} & \frac{\delta f_3}{\delta x_5} & \frac{\delta f_3}{\delta x_6} & \frac{\delta f_3}{\delta x_7} & \frac{\delta f_3}{\delta x_8} & \frac{\delta f_3}{\delta x_9} \\ 0 & 0 & 0 & \frac{\delta f_4}{\delta x_4} & 0 & 0 & \frac{\delta f_4}{\delta x_7} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\delta f_5}{\delta x_5} & 0 & 0 & \frac{\delta f_5}{\delta x_8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\delta f_6}{\delta x_6} & 0 & 0 & \frac{\delta f_6}{\delta x_9} \\ \frac{\delta f_7}{\delta x_1} & \frac{\delta f_7}{\delta x_2} & \frac{\delta f_7}{\delta x_3} & \frac{\delta f_7}{\delta x_4} & \frac{\delta f_7}{\delta x_5} & 0 & \frac{\delta f_7}{\delta x_7} & \frac{\delta f_7}{\delta x_8} & 0 \\ 0 & \frac{\delta f_8}{\delta x_2} & \frac{\delta f_8}{\delta x_3} & \frac{\delta f_8}{\delta x_4} & 0 & 0 & \frac{\delta f_8}{\delta x_7} & 0 & 0 \\ 0 & \frac{\delta f_9}{\delta x_2} & \frac{\delta f_9}{\delta x_3} & \frac{\delta f_9}{\delta x_4} & \frac{\delta f_9}{\delta x_5} & 0 & \frac{\delta f_9}{\delta x_7} & \frac{\delta f_9}{\delta x_8} & 0 \end{bmatrix} \tag{1.3}$$

Some useful properties

Using the chain rule, we have

$$\frac{\delta y}{\delta \dot{\theta}} = \frac{\delta y}{\delta t} \times \frac{\delta t}{\delta \dot{\theta}} \tag{1.4}$$

which provides the following outcome:

$$\frac{\delta y}{\delta \dot{\theta}} = \frac{\delta y}{\delta t} \times (\frac{\delta \dot{\theta}}{\delta t})^{-1} = \frac{\delta y}{\delta t} \times (\frac{\delta^2 \theta}{\delta t^2})^{-1} = \frac{\dot{y}}{\ddot{\theta}} \tag{1.5}$$

Also if

$$F = f(a(t), b(t)) \tag{1.6}$$

where F does not depend directly on t, then the second outcome (using the total derivative rule) is:

$$\frac{dF}{dt} = \frac{\delta f}{\delta a}\frac{da}{dt} + \frac{\delta f}{\delta t}\frac{db}{dt} \tag{1.7}$$

Deriving the elements of the Jacobian

Bear in mind that the following differentiations are for a discrete process, t has been used for familiarity's sake, and only at points to remind us that all the variables are time based. It is also assumed that the signal dynamics are varying sufficiently slowly between samples that the system can be assumed continuous in terms of differentiation. Also the sampling, with a rate of dt is of a signal containing frequencies below $\frac{2\pi}{dt}$ so that the continuous signal can be reconstructed (according to the Nyquist theorem). See [?] for situations where this is not the case.

The first differential follows along the same lines as Equation 1.5:

$$\frac{\delta f_1}{\delta x_4} = \frac{\delta}{\delta \phi}(\dot{\phi} - \dot{\psi} sin\theta) = \frac{\delta}{\delta \phi}(\dot{\phi}) = \frac{\delta \dot{\phi}}{\delta t} \times \frac{\delta t}{\delta \phi} = \ddot{\phi} \times \dot{\phi}^{-1} = \ddot{\phi}/\dot{\phi} \tag{1.8}$$

$$\frac{\delta f_1}{\delta x_5} = \frac{\delta}{\delta \theta}(\dot{\phi} - \dot{\psi} sin\theta) = -\dot{\psi} cos\theta \tag{1.9}$$

In a similar fashion to Equation 1.8 we get:

$$\frac{\delta f_1}{\delta x_6} = \frac{\delta}{\delta \psi}(\dot{\phi} - \dot{\psi} sin\theta) = -(\ddot{\psi}/\dot{\psi}) sin\theta \tag{1.10}$$

$$\frac{\delta f_1}{\delta x_7} = \frac{\delta}{\delta \dot{\phi}}(\dot{\phi} - \dot{\psi} sin\theta) = 1 \tag{1.11}$$

The next equation is initially expanded based on Equation 1.4 and later the total derivative rule (Equation 1.7) is used:

$$\begin{aligned}
\frac{\delta f_1}{\delta x_8} &= \frac{\delta}{\delta \dot{\theta}}(\dot{\phi} - \dot{\psi} sin\theta) = \frac{\delta}{\delta \dot{\theta}}(-\dot{\psi} sin\theta) = \frac{\delta}{\delta t}(-\dot{\psi} sin\theta) \times \frac{\delta t}{\delta \dot{\theta}} \\
&= \frac{\delta}{\delta t}(-\dot{\psi}(t) sin\theta(t)) \times (\frac{\delta \dot{\theta}}{\delta t})^{-1} \\
&= \left[\frac{\delta}{\delta \dot{\psi}}(-\dot{\psi}(t) sin\theta(t)) \frac{d\dot{\psi}}{dt} + \frac{\delta}{\delta \theta}(-\dot{\psi}(t) sin\theta(t)) \frac{d\theta}{dt} \right] \times \frac{1}{\ddot{\theta}} \\
&= \left[-(sin\theta)\ddot{\psi} - (\dot{\psi} cos\theta)\dot{\theta} \right] \times \frac{1}{\ddot{\theta}}
\end{aligned} \tag{1.12}$$

$$\frac{\delta f_1}{\delta x_9} = \frac{\delta}{\delta \dot{\psi}}(\dot{\phi} - \dot{\psi} sin\theta) = -sin\theta \tag{1.13}$$

$$\frac{\delta f_2}{\delta x_4} = \frac{\delta}{\delta \phi}(\dot{\theta} cos\theta + \dot{\psi} cos\theta sin\phi) = \dot{\psi} cos\theta cos\phi \tag{1.14}$$

For this next differential, the product rule followed by the result of Equation 1.8 is used:

$$\begin{aligned}
\frac{\delta f_2}{\delta x_5} &= \frac{\delta}{\delta \theta}(\dot{\theta} cos\theta + \dot{\psi} cos\theta sin\phi) \\
&= \left[\dot{\theta}\frac{\delta}{\delta \theta}(cos\theta) + cos\theta\frac{\delta}{\delta \theta}(\dot{\theta})\right] - \dot{\psi} sin\theta sin\phi \\
&= -\dot{\theta} sin\theta + \frac{\ddot{\theta}}{\dot{\theta}} cos\theta - \dot{\psi} sin\theta sin\phi
\end{aligned} \tag{1.15}$$

$$\frac{\delta f_2}{\delta x_6} = \frac{\delta}{\delta \psi}(\dot{\theta} cos\theta + \dot{\psi} cos\theta sin\phi) \tag{1.16}$$

The following uses the outcome of Equation 1.5:

$$\begin{aligned}
\frac{\delta f_2}{\delta x_7} &= \frac{\delta}{\delta \dot{\theta}}(\dot{\theta} cos\theta + \dot{\psi} cos\theta sin\phi) \\
&= \left[\dot{\theta}\frac{\delta}{\delta \dot{\theta}}(cos\theta) + cos\theta\frac{\delta}{\delta \dot{\theta}}(\dot{\theta})\right] - \dot{\psi} sin\theta sin\phi \\
&= \dot{\theta}\frac{1}{\ddot{\theta}}\frac{\delta}{\delta t}(cos\theta(t)) + cos\theta - \dot{\psi} sin\theta sin\phi
\end{aligned} \tag{1.17}$$

$$\frac{\delta f_2}{\delta x_8} = \frac{\delta}{\delta \dot{\psi}}(\dot{\theta} cos\theta + \dot{\psi} cos\theta sin\phi) \tag{1.18}$$

$$\frac{\delta f_2}{\delta x_9} = \frac{\delta}{\delta \dot{\phi}}(\dot{\theta} cos\theta + \dot{\psi} cos\theta sin\phi) \tag{1.19}$$

$$\frac{\delta f_3}{\delta x_4} = \frac{\delta}{\delta \phi}(-\dot{\theta} sin\theta + \dot{\psi} cos\theta cos\phi) \tag{1.20}$$

$$\frac{\delta f_3}{\delta x_5} = \frac{\delta}{\delta \theta}(-\dot{\theta} sin\theta + \dot{\psi} cos\theta cos\phi) \tag{1.21}$$

$$\frac{\delta f_3}{\delta x_6} = \frac{\delta}{\delta \psi}(-\dot{\theta} sin\theta + \dot{\psi} cos\theta cos\phi) \tag{1.22}$$

$$\frac{\delta f_3}{\delta x_7} = \frac{\delta}{\delta \dot{\phi}}(-\dot{\theta} sin\theta + \dot{\psi} cos\theta cos\phi) \tag{1.23}$$

$$\frac{\delta f_3}{\delta x_8} = \frac{\delta}{\delta \dot{\theta}}(-\dot{\theta} sin\theta + \dot{\psi} cos\theta cos\phi) \tag{1.24}$$

$$\frac{\delta f_3}{\delta x_9} = \frac{\delta}{\delta \dot{\psi}}(-\dot{\theta} sin\theta + \dot{\psi} cos\theta cos\phi) = cos\theta cos\phi \tag{1.25}$$

In the following six equations, dt is the sample time, which is treated as a constant.

$$\begin{aligned} \frac{\delta f_4}{\delta x_4} &= \frac{\delta}{\delta \phi}(\phi + \dot{\phi} dt) \\ &= 1 + \frac{\ddot{\phi}}{\dot{\phi}} dt \end{aligned} \tag{1.26}$$

$$\begin{aligned} \frac{\delta f_4}{\delta x_7} &= \frac{\delta}{\delta \dot{\phi}}(\phi + \dot{\phi} dt) \\ &= \frac{\dot{\phi}}{\ddot{\phi}} + dt \end{aligned} \tag{1.27}$$

$$\frac{\delta f_5}{\delta x_5} = \frac{\delta}{\delta \theta} \tag{1.28}$$

$$\frac{\delta f_5}{\delta x_8} = \frac{\delta}{\delta \dot{\theta}} \tag{1.29}$$

$$\frac{\delta f_6}{\delta x_6} = \frac{\delta}{\delta \psi} \tag{1.30}$$

$$\frac{\delta f_6}{\delta x_9} = \frac{\delta}{\delta \dot{\psi}} \tag{1.31}$$

$$\frac{\delta f_7}{\delta x_1} = \frac{\delta}{\delta p} \tag{1.32}$$

$$\frac{\delta f_7}{\delta x_2} = \frac{\delta}{\delta q} \tag{1.33}$$

$$\frac{\delta f_7}{\delta x_3} = \frac{\delta}{\delta r} \tag{1.34}$$

$$\frac{\delta f_7}{\delta x_4} = \frac{\delta}{\delta \phi} \tag{1.35}$$

$$\frac{\delta f_7}{\delta x_5} = \frac{\delta}{\delta \theta} \tag{1.36}$$

$$\frac{\delta f_7}{\delta x_7} = \frac{\delta}{\delta \dot{\phi}} \tag{1.37}$$

$$\frac{\delta f_7}{\delta x_8} = \frac{\delta}{\delta \dot{\theta}} \tag{1.38}$$

$$\frac{\delta f_8}{\delta x_2} = \frac{\delta}{\delta q} \tag{1.39}$$

$$\frac{\delta f_8}{\delta x_3} = \frac{\delta}{\delta r} \tag{1.40}$$

$$\frac{\delta f_8}{\delta x_4} = \frac{\delta}{\delta \phi} \tag{1.41}$$

$$\frac{\delta f_8}{\delta x_7} = \frac{\delta}{\delta \dot{\phi}} \tag{1.42}$$

$$\frac{\delta f_9}{\delta x_5} = \frac{\delta}{\delta} \tag{1.43}$$

$$\frac{\delta f_9}{\delta x_2} = \frac{\delta}{\delta q} \tag{1.44}$$

$$\frac{\delta f_9}{\delta x_3} = \frac{\delta}{\delta r} \tag{1.45}$$

$$\frac{\delta f_9}{\delta x_4} = \frac{\delta}{\delta \phi} \tag{1.46}$$

$$\frac{\delta f_9}{\delta x_5} = \frac{\delta}{\delta \theta} \tag{1.47}$$

$$\frac{\delta f_9}{\delta x_7} = \frac{\delta}{\delta \dot{\psi}} \tag{1.48}$$

$$\frac{\delta f_9}{\delta x_8} = \frac{\delta}{\delta \dot{\theta}} \tag{1.49}$$