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A DISCRETE SINGULARITY FORMULATION FOR SOLVING STEADY, FULLY DEVELOPED DUCT FLOWS

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SUMMARY

A very effective numerical formulation is proposed to solve approximately the problem of steady, fully developed laminar flows of incompressible fluids in straight ducts of arbitrary, simply connected cross-sections. The method is an application of the discrete singularity technique, with a finite number of singularities located outside the domain concerned. To illustrate the usefulness of the method, some examples of computations are also presented.

INTRODUCTION

The problem of steady, fully developed laminar flows of incompressible fluids in arbitrarily shaped constant ducts is rather primitive in fluid mechanics, but very important in the field of practical engineering. This classical problem, which is fundamentally governed by the Laplace equation, as will be mentioned later, has hitherto been solved by a variety of techniques such as complex variable methods with conformal transformations reviewed by Happel[1] and White,[2] and boundary residual methods with harmonic functions employed by Sparrow and Haji-Shiekh, Shih,[4] Ratkowsky[5] and others. In general, however, these techniques are rather limited in applicability.

The present approach is based on the discrete singularity method first proposed by Mathon and Johnston[6] to solve general elliptic boundary value problems, and then applied by the present authors[7] to solve two-dimensional low-Reynolds-number flows. It basically falls within the boundary residual method with harmonic functions, and is superior in both applicability and simplicity for computer simulations.

APPROXIMATE METHOD OF SOLUTION

As is well known, fully developed laminar flows of incompressible fluids in straight ducts are governed by

$$\nabla^2 u = \frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z} \tag{1}$$

with

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

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where p designates the pressure, μ the viscosity, (x, y, z) THE Cartesian co-ordinates with z being in the flow direction, and finally u represents the velocity component along z. With

$$u = \psi + \frac{1}{4\mu}\frac{dp}{dz}(x^2 + y^2) \tag{2}$$

equation (1) leads to

$$\nabla^2 \psi = 0 \tag{3}$$

In this case, the subjected boundary condition is the so-called no-slip condition represented by

$$\psi = -\frac{1}{4\mu}\frac{dp}{dz}(x^2 + y^2) \quad \text{on } \Gamma \tag{4}$$

where Γ is the boundary.

According to Mathon and Johnston,[6] a fundamental solution of equation (3) has the form

$$\psi_j = \log r_j \tag{5}$$

with

$$r_j = \sqrt{[(x - \xi_j)^2 + (y - \eta_j)^2]}$$

where (ξ_j, η_j) designate the co-ordinates of a singularity T_j.

Then, we assume an approximation $\hat{u}_n$ to the solution u in the following form:

$$\hat{u}_n = \sum_{j=1}^{n} c_j \log r_j + \frac{1}{4\mu}\frac{dp}{dz}(x^2 + y^2) \tag{6}$$

where the positions of singularities T_j which define r_j, and the parameters c_j are both unknown. Since equation (6) satisfies the governing equation (1), the unknowns in (6) can be determined by the least square boundary residual technique. Thus we can minimize the integral

$$I = \oint_\Gamma \hat{u}_n^2 \, ds \tag{7}$$

where s denotes a curvilinear co-ordinate along Γ.

This nonlinear minimization problem is much simplified by assuming that n singularities $T_1, T_2, \ldots, T_n$ are discretely distributed on a circle of unknown radius R which encloses the domain, as is illustrated in Figure 1. In this case, the centre of the circle is so located that the radius R can vary to the minimum.

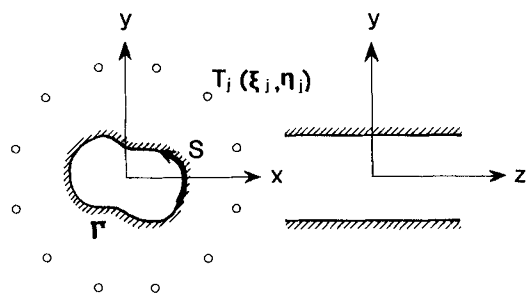

Figure 1. Co-ordinate system

With a given R, we have

$$\frac{\partial I}{\partial c_i}=0 \qquad i=1,2,\ldots,n \tag{8}$$

which leads to

$$[a]\{c\}=\{d\} \tag{9}$$

where

and

$$\left.\begin{aligned} a_{ij} &= \oint_\Gamma \log r_i \log r_j \, \mathrm{d}s \\ d_i &= -\frac{1}{4\mu}\frac{\mathrm{d}p}{\mathrm{d}z}\oint_\Gamma (x^2+y^2)\log r_i \, \mathrm{d}s \end{aligned}\right\} \qquad i,j=1,2,\ldots,n$$

These elements a_{ij} and d_i can be easily obtained by numerical integrations with the trapezoidal rule. Then, equation (9) can be solved numerically with a computer. The optimum value of R which minimizes I, is determined from the above computations with various values of R.

In addition, the flow rate Q through the duct in question is written as

$$Q=\int_\Omega \hat{u}_n \, \mathrm{d}\Omega \tag{10}$$

where Ω denotes the domain or duct sectional area. With the aid of equation (6), it is easy to show that

$$Q=Q_0+\sum_{j=1}^{n} c_j Q_j \tag{11}$$

$$Q_0=\frac{1}{16\mu}\frac{\mathrm{d}p}{\mathrm{d}z}\oint_\Gamma (x^2+y^2)^2\frac{\mathrm{d}\theta}{\mathrm{d}s}\mathrm{d}s \tag{12}$$

and

$$Q_j=\frac{1}{4}\oint_\Gamma r_j^2(2\log r_j-1)\frac{\mathrm{d}\theta_j}{\mathrm{d}s}\mathrm{d}s \tag{13}$$

with

$$\theta=\tan^{-1}\frac{y}{x} \qquad \theta_j=\tan^{-1}\frac{y-\eta_i}{x-\xi_j}$$

With these equations, the surface integral (10) can be reduced to the curvilinear integrals.

NUMERICAL DISCUSSIONS

First, to clarify the characteristics of the present method, we examined the behaviour of the boundary residuals for the case of rectangular ducts of width $2a$ and height $2b$, with the aspect ratio β defined by $\beta=b/a$. Figure 2 shows ϕ-dependence of $\bar{\varepsilon}$ for $n=8$, where

$$\phi=\frac{R}{\sqrt{(a^2+b^2)}} \tag{14}$$

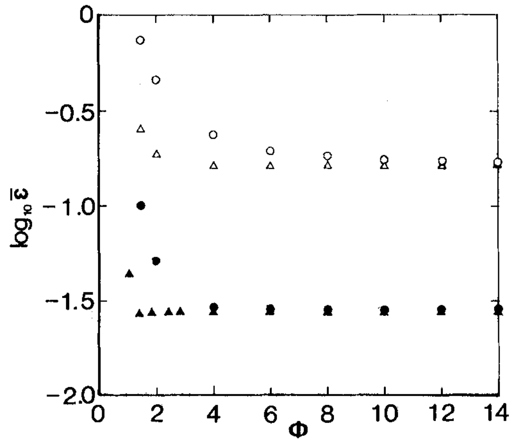

Figure 2. ϕ-dependence of $\bar{\varepsilon}$ with $n = 8$ in the case of rectangular ducts: △ $\beta = 1/3$, ▲ $\beta = 1$ (with circular distributions); ○ $\beta = 1/3$, ● $\beta = 1$(with similar distributions)

which specifies the distribution of singularities, and $\bar{\varepsilon}$ is defined as

$$\bar{\varepsilon} = \frac{1}{u_m}\sqrt{\left(\frac{I}{s_0}\right)} \tag{15}$$

where u_m and s_0 denote the mean velocity and the duct perimeter, respectively. For comparison, we added in this figure the data for another mode of singularity distribution on a closed curve similar to Γ, where ϕ is redefined as a similarity ratio between the curve and Γ. Obviously, there is a general tendency that $\bar{\varepsilon}$ value first decreases monotonically with increasing ϕ, and then becomes almost constant at very high values of ϕ. And so, we can choose a proper smaller value of ϕ as an optimum one. Therefore, the circular distribution is considered superior for its simplicity.

Figure 3 shows the computed iso-velocity lines for the case of $\beta = 1/5$, $n = 16$. This result agrees very well with the exact solution, although the evidence is omitted here.

Furthermore, we computed the products of Fanning friction factor f and Reynolds number (now denoted by $f\cdot\text{Re}$ values) for some kinds of duct geometries.

The definitions of f and Re are as follows:

$$f = -\frac{2r_h}{\rho u_m^2}\frac{dp}{dz}$$

where ρ denotes the fluid density and

$$\text{Re} = \frac{4r_h u_m}{\nu} \qquad r_h = \frac{A}{s_0}$$

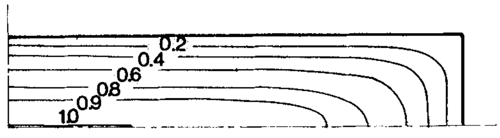

Figure 3. Computed iso-velocity lines in u/u_{max} in a case of a rectangular duct: $\beta = 1/5$, $n = 16$

where ν is the kinematic viscosity of the fluid and A the area of the domain or sectional area of the duct. Using the trapezoidal rule with 100 divisions for boundary integrals, we obtained the $f \cdot \mathrm{Re}$ values with circular distributions of eight singularities ($n = 8$), in the cases of rectangular, elliptical and composite ducts, the last of which has one pair of opposing walls and the other pair of semi-circular arcs, as shown in Figure 4. Such a duct geometry has already been treated by Zarling[8] by means of his technique which combines Schwarz–Neumann alternating method and the least square point matching formulation. Figure 4 indicates the present results of $f \cdot \mathrm{Re}$ values against the aspect ratios β (whose definitions are presented in the figure), in comparison with the exact solution of rectangular geometry after Cornish,[9] that of elliptical geometry after Berker[10] and approximate solution of composite geometry after Zarling.[8] The agreements of the present solutions with the others are very good. Considering that the other methods are generally sophisticated, the proposed method can be regarded as an excellently efficient approach for solving such duct flow problems. It should be added that less than 1 sec is required to compute each $f.\mathrm{Re}$ value in Figure 4 on a FACOM M-200 computer in double precision (64 bits per floating point word), and less than 4 min is required for the same purpose on a DEC11 mini-computer. However, the computing time depends largely on the number of divisions in numerical integrations required to obtain a_{ij} and d_i in equation (9). And it was clarified from various simulations that the division number of 100 (which is our standard value) is sufficiently large for the present duct geometries.

Lastly, we tried to propose data for a rough error estimation of computed $f \cdot \mathrm{Re}$ values. Figure 5 shows the relation between $|e|$ and $\bar{\varepsilon}$ for various rectangular ducts with $n = 8$, 16 and 32, where

$$e = \{f \cdot \mathrm{Re} - (f \cdot \mathrm{Re})_{ex}\}/(f \cdot \mathrm{Re})_{ex}$$

with $(f \cdot \mathrm{Re})_{ex}$ being the exact solution of $f \cdot \mathrm{Re}$ value obtained after Cornish.[9] Obviously, there is a vague relation between $|e|$ and $\bar{\varepsilon}$. And it should be noted that the following inequality

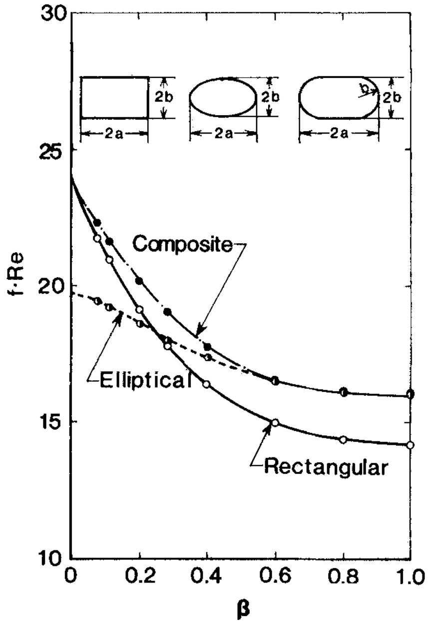

Figure 4. Relations between $f.\mathrm{Re}$ value and aspect ratio $\beta = b/a$ for various ducts: ○, ◑, ● present results; —— exact solution after Cornish;[9] - - - - - exact solution after Berker;[10] —·— approximate solution after Zarling[8]

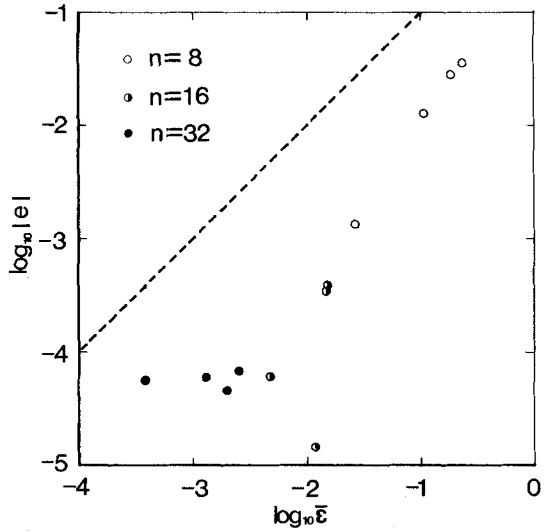

Figure 5. Relation between $|e|$ and $\bar{\varepsilon}$ for various rectangular ducts: $n = 8, 16, 32$; $\beta = 1/4, 1/3, 1/2, 1$; ----- bound of the inequality (16)

holds, as shown by a dotted line in Figure 5:

$$|e| < \bar{\varepsilon} \tag{16}$$

Therefore, it can serve for a rough estimation of the orders of errors in computed $f \cdot \mathrm{Re}$ values for duct geometries other than rectangular ones which are not so complicated, when the $\bar{\varepsilon}$ values are known.

CONCLUSIONS

A very efficient computational technique was presented to solve the problem of steady laminar flows of incompressible fluids in straight ducts of arbitrary, simply connected cross-sections. It is based on a discrete singularity method with a circular distribution. Simple numerical simulations show a good effectiveness of the method.

Additionally, it is necessary to confirm the range of applicability with many numerical simulations for various ducts of complicated shapes.

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