Calculating the Number of Goldbach Partitions

An alternative approach

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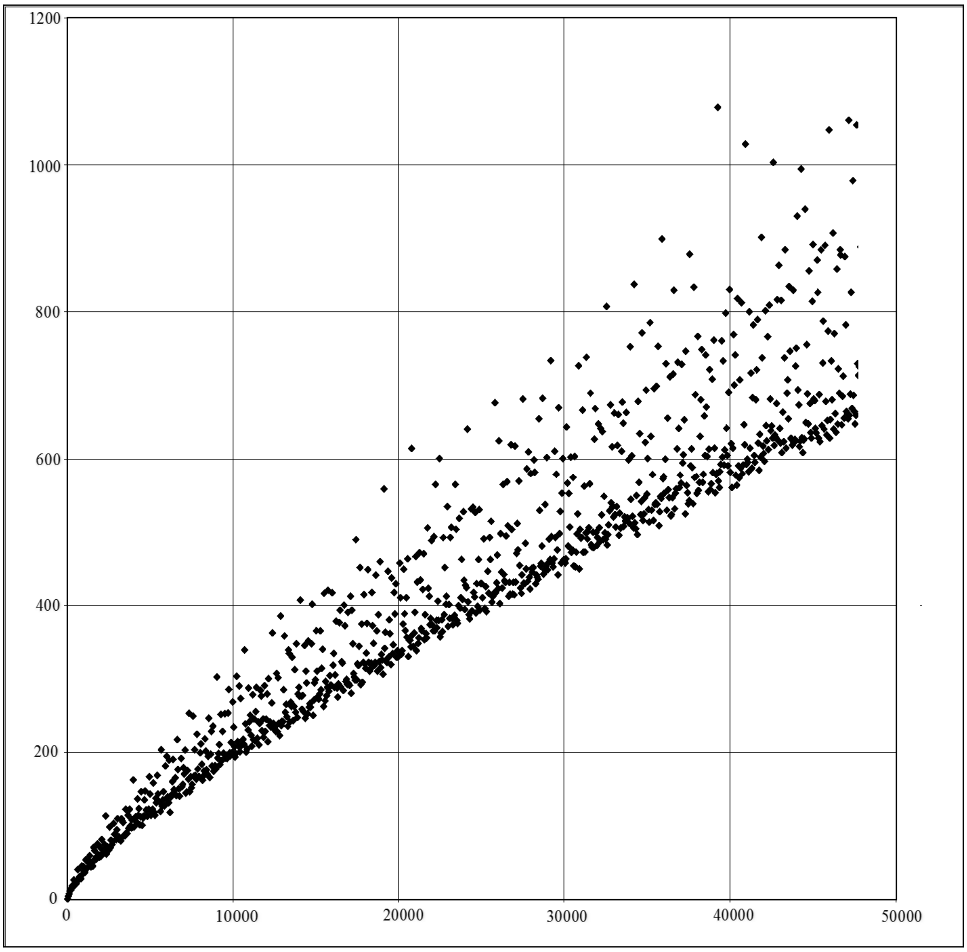
Abstract

The sum oft two odd primes is an even number. The inverse proposition (any even number $n \geq 6$ can be written in at least one way as a sum of two odd primes) has been the subject of a correspondence between Goldbach and Euler, in 1742. It has never been proved, nor disproved.

As n grows larger, the number of possible partitions, $g(n)$, increases in an irregular way, depending heavily on n's divisors. A plot of $g(n)$ reveals a comet-like pattern as shown on Figure 1.

Fig. 1 Plot of $g(n)$ for values $n = 6 + m \cdot 48$; $(1 \leq m \leq 1000)$



Asymptotic formulas, describing $g(n)$'s dependency on *n*'s divisors, have been conjectured by a number of authors. Most formulas contain the factors, which had been identified by Sylvester as far as in 1871.

In this paper, we present a new approach, yielding fairly accurate results, which are compared with those obtained by known formulas.

The Hardy-Littlewood conjecture

The work of Brun(1915), Stäckel (1916-1918), Hardy and Littlewood (1922), among others, confirmed a formula, which is equivalent to the one which Sylvester had arrived at:

$$g(n) \approx C \cdot \frac{n}{\log^2(n)} \cdot \prod_{\substack{3<p<\sqrt{n} \\ p/n}} \frac{(p-1)}{(p-2)}, \tag{1.1}$$

$g(n)$ representing the count of partitions into prime numbers q and $n-q$, with $q \le n/2$, and the product being taken over the odd prime divisors of n.

The approach taken by British mathematicians Hardy and Littlewood $[1]$ is interesting. Dealing with the general problem of partitioning numbers into primes, they had developed their *circle method*, which enabled them to prove, that *every sufficiently large even number is the sum of four odd prime numbers, and every sufficiently large odd number the sum of three*. If, in the case of two primes, the method failed to yield a proof, it nevertheless helped to support Sylvesters formula. Moreover, reviewing the different values, which had been proposed for the constant factor C (Sylvester had taken $C = 0.5$), they found a strong argument for it being equal to the “twin primes constant”:

$$C_{HL} = \prod_{3 \le p \le \infty} \frac{p \cdot (p-2)}{(p-1)^2} = 0.6601618158\ldots, \tag{1.2}$$

An alternative formula

From the $\pi(n)$ prime numbers, which are inferior to n , we sieve out those, which do not lend themselves to a Goldbach partition. This approach implies multiplying $\pi(n)$ by a factor. To support our alternative formula:

$$g_{alt}(n) \approx \frac{1}{2} \cdot \pi(n) \cdot \prod_{\substack{3<p<\sqrt{n} \\ p \times n}} \frac{(p-2)}{(p-1)} \tag{2}$$

(the product being taken over the primes relative to n), we present a heuristic argumentation as follows:

Asymptotically, the count of integers, which contain none of the factors p $\left(3 \le p \le p_m\right)$, can be estimated with the expression

$$m \cdot \prod_{3 \le p \le p_m} \frac{p-1}{p} \; ; \qquad (m \to \infty)$$

Thus, we may write, for the count prime numbers q $\left(\sqrt{n} < q < n\right)$:

$$\pi(n) - \sqrt{n} \approx n \cdot \prod_{3 \le p < \sqrt{n}} \frac{p-1}{p} \qquad (n \to \infty) \text{ ,and even}$$

$$\frac{\pi(n)}{n} \approx \prod_{3 \le p < \sqrt{n}} \frac{p-1}{p} \qquad \text{and} \qquad \frac{1}{\log(n)} \approx \prod_{3 \le p < \sqrt{n}} \frac{p-1}{p} \qquad (2.1)$$

Prime numbers, which participate in a Goldbach partition $q_i + q_k = n$ must satisfy the reflexive conditions :

$$q_i (\text{mod } p) \ne n (\text{mod } p) \quad \Leftrightarrow \quad q_k (\text{mod } p) \ne n (\text{mod } p) \quad ; \; (3 \le p < \sqrt{n}\,) \qquad (2.2.1)$$

Asymptotically, the count of such prime numbers q_i can be estimated with the expression

$$G(n) \approx n \cdot \prod_{\substack{3 < p < \sqrt{n} \\ p/n}} \frac{p-1}{p} \cdot \prod_{\substack{3 < p < \sqrt{n} \\ p \times n}} \frac{p-2}{p} \qquad (n \to \infty) \text{ , or}$$

$$G(n) \approx \pi(n) \cdot \prod_{\substack{3 \le p < \sqrt{n} \\ p/n}} (p-1) \cdot \prod_{\substack{3 \le p < \sqrt{n} \\ p \times n}} (p-2) \Bigg/ \prod_{3 \le p < \sqrt{n}} (p-1) \qquad (2.2.2)$$

Simplifying the quotient, and counting partitions into prime numbers q and $n-q$, with $q \le n/2$, we obtain the formula (2).

Asymptotically, the formulas (1.1, 1.2), and (2) are equivalent

With (2.1, 2.2) we find

$$\frac{g_{alt}}{g_{HL}} = \left[\frac{\pi(n)}{n/\log(n)}\right] \cdot \left[\frac{1}{1/\log(n)} \cdot \prod_{2 \le p < \sqrt{n}} \frac{(p-1)}{p}\right] \cdot \left[\prod_{\sqrt{n} < p \le \infty} \frac{(p-1)^2}{p \cdot (p-2)}\right] \approx 1,$$

each of the three factors in square brackets tending towards unity.

Results

In their paper $[1]$, Hardy and Littlewood state, that, in the formula (1.1), the term $\log^2(n)$ of the denominator "is certainly in error wrong to an order $\log(n)$" and, as had been suggested by Shah and Wilson, must be replaced by $\log^2(n)-2\cdot\log(n)$, a substitution, which is asymptotically irrelevant, but essential for the purpose of verification within the limits of calculation.

Results computed with the formulas

$$g_{SW}(n)\approx C_{HL}\cdot\frac{n}{(\log^2 n-2\cdot\log n)}\cdot\prod_{\substack{3\le p\le\sqrt{n}\\ p/n}}\frac{(p-1)}{(p-2)} \tag{1.3}$$

respectively

$$g_{alt}(n)\approx\frac{1}{2}\cdot\pi(n)\cdot\prod_{\substack{3<p<\sqrt{n}\\ p\times n}}\frac{(p-2)}{(p-1)}, \tag{2}$$

are compared on Fig.2, and on Table 1.

Fig. 2. Calculated results $g_{alt}(n)$ and $g_{SW}(n)$ compared with $g(n)$

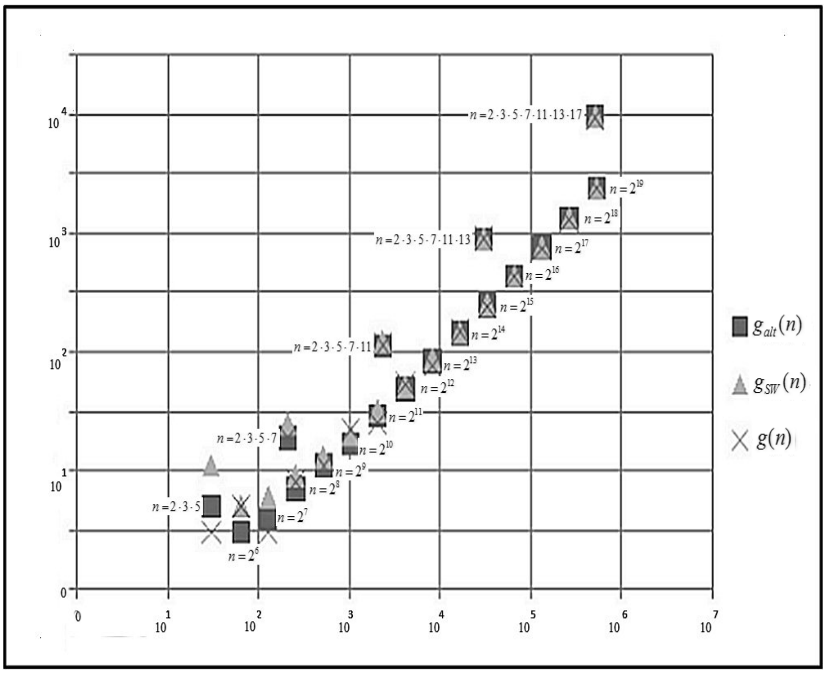


Table 1. Calculated results $g_{alt}(n)$ and $g_{SW}(n)$ compared with $g(n)$

n	$\pi(n)$	$g(n)$	$g_{alt}(n)$	$g_{SW}(n)$	g_{alt}/g	g_{SW}/g
2^{6} = 64	18	5	3	5	0,600	1,000
2^{7} = 128	31	3	4	6	1,333	2,000
2^{8} = 256	54	8	7	9	0,875	1,125
2^{9} = 512	97	11	11	13	1,000	1,182
2^{10} = 1024	172	22	17	20	0,773	0,909
2^{11} = 2048	309	25	29	32	1,160	1,280
2^{12} = 4096	564	53	49	51	0,925	0,962
2^{13} = 8192	1028	76	83	86	1,092	1,132
2^{14} = 16384	1900	151	143	145	0,947	0,960
2^{15} = 32768	3512	244	246	248	1,008	1,016
2^{16} = 65536	6542	435	434	429	0,998	0,986
2^{17} = 131072	12251	749	765	751	1,021	1,003
2^{18} = 262144	23000	1314	1356	1324	1,032	1,008
2^{19} = 524288	43390	2367	2431	2353	1,027	0,994
2*3*5 = 30	10	3	5	11	1,667	3,667
2*3*5*7 = 210	46	19	19	25	1,000	1,316
2*3*5*7*11 = 2310	343	114	112	122	0,982	1,070
2*3*5*7*11*13 = 30030	3248	905	893	898	0,987	0,992
2*3*5*7*11*13*17 = 510510	42331	9493	9827	9521	1,035	1,003
2*3*5*7*11*17*19*23*29 = 497668710	26239628	3977551*)	4228577	3970818	1,063	0,998

*) For $6 \le n \le 5 \cdot 10^{8}$, the maximal value assumed by $g(n)$ (Richstein[3])

Discussion

For its simplicity, the formula $g_{alt}(n)$ adequately describes the behaviour of $g(n)$, even though it does not quite attain the degree of precision granted by $g_{SW}(n)$. The latter seems to be a remarkably accurate formula, indeed. After testing several formulas up to $n \le 5 \cdot 10^8$, Richstein $[3]$ finds it "a bit surprising" , that $g_{SW}(n)$ yields better results, on the whole, than other, more elaborate ones (with the exception of Selmer's formulas).

References

[1] G.H.Hardy, J.E. Littlewood, *Some problems of 'partitio numerorum'; III: On the expression of a number as a sum of primes*, Acta Math. **44** (1922), 32 – 39

[2] Wang Yuan (Ed.), *Goldbach Conjecture*, 1984, World Scientific Publishing, Singapore, ISBN 9971-966-08-5, and ISBN 9971-966-09-3 pbk (a collection of selected papers, containing a photographic reproduction of [1])

[3] Richstein J. ; Bosma Wieb (Editor*); Computing the number of Goldbach partitions up to 5.10^8*, Conference : Algorithmic number theory. International symposium, 4, Leiden, NLD, 2000-07-02 , ISSN : 0302-9743 ISBN : 3-540-67695-3