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Estimating the Number of Goldbach Partitions

An alternative approach

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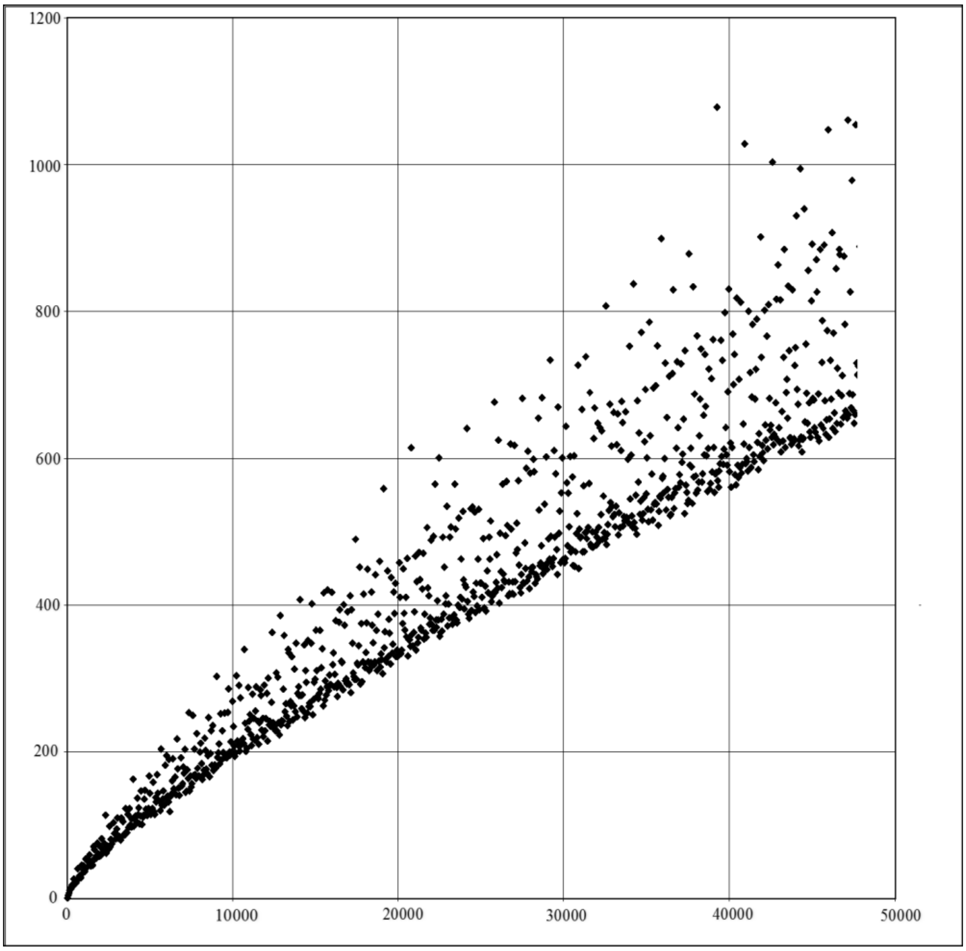
Abstract

The sum oft two odd primes is an even number. The inverse proposition (any even number $n \geq 6$ can be written in at least one way as a sum of two odd primes) has been the subject of a correspondence between Goldbach and Euler, in 1742. It has never been proved, nor disproved.

As n grows larger, the number of possible partitions, $g(n)$, increases in an irregular way, depending heavily on n's divisors. A plot of $g(n)$ reveals a comet-like pattern as shown on **Figure 1**.

Fig. 1 Plot of $g(n)$ for values $n = 6 + m \cdot 48$; $(1 \leq m \leq 1000)$



Asymptotic formulas, describing $g(n)$'s dependency on *n*'s divisors, have been conjectured by a number of authors. Most formulas contain the factors, which had been identified by Sylvester as far as in 1871.

In this paper, we present a new approach, yielding fairly accurate results, which are compared with those obtained by known formulas.

The Hardy-Littlewood conjecture

The work of Brun(1915), Stäckel (1916-1918), Hardy and Littlewood (1922), among others, confirmed a formula, which, essentially, is equivalent to the one which Sylvester had arrived at:

$$g(n) \sim C \cdot \frac{n}{\log^2(n)} \cdot \prod_{\substack{3<p<\sqrt{n} \\ p/n}} \frac{p-1}{p-2}, \qquad (1.1)$$

$g(n)$ representing the count of partitions into prime numbers q and $n-q$, with $q \le n/2$, and the product being taken over the odd prime divisors of n.

The approach taken by British mathematicians Hardy and Littlewood $[1]$ is interesting. Dealing with the general problem of partitioning numbers into primes, they had developed their *circle method*, which enabled them to prove, that *every sufficiently large even number is the sum of four odd prime numbers, and every sufficiently large odd number the sum of three*. If, in the case of two primes, the method failed to yield a proof, it nevertheless helped to support Sylvesters formula. Moreover, reviewing the different values, which had been proposed for the constant factor C (Sylvester had taken $C = 0.5$), they found a strong argument for it being equal to the “twin primes constant”:

$$C_{HL} = \prod_{3 \le p \le \infty} \frac{p \cdot (p-2)}{(p-1)^2} = 0.6601618158..., \qquad (1.2)$$

An alternative formula

From the $\pi(n)$ prime numbers, which are inferior to n, we sieve out those, which do not lend themselves to a Goldbach partition. This approach implies multiplying $\pi(n)$ by a factor :

$$g_{alt}(n) \sim \frac{1}{2} \cdot \pi(n) \cdot \prod_{\substack{3<p<\sqrt{n} \\ p \times n}} \frac{p-2}{p-1}, \qquad (2a)$$

the product being taken over the prime numbers p *which are primes relative to* n :
$n \ne 0 \pmod p$.

To support this formula we use Merten's 3rd theorem:

$\lim_{n\to\infty} \prod_{2\le p\le n} \frac{p-1}{p} \cdot \log(n) = e^{-\gamma}$, with the Euler-Mascheroni constant $\gamma = 0.5772156649...$,

from which we deduce the asymptotic equivalences

$$\frac{1}{2} \cdot e^{\gamma} \cdot \prod_{2\le p<\sqrt{n}} \frac{p-1}{p} \sim \frac{1}{\log(n)}, \quad \text{and} \quad \frac{\pi(n)}{n} \sim \frac{1}{2} \cdot e^{\gamma} \cdot \prod_{2\le p<\sqrt{n}} \frac{p-1}{p} \qquad (2.1)$$

Numbers q ($\sqrt{n} < q < n$) which lend themselves to a Goldbach Partition must satisfy the two conditions:

$n \ne 0 \pmod{p}$ and $n \ne q \pmod{p}$ for all prime numbers p ($2 \le p < \sqrt{n}$) (2.2.1)

To estimate their density, we have to modify the formula (2.1) to :

$$\frac{G(n)}{n} \sim \frac{1}{2} \cdot e^{\gamma} \cdot \prod_{\substack{2\le p<\sqrt{n} \\ p/n}} \frac{p-1}{p} \cdot \prod_{\substack{3\le p<\sqrt{n} \\ p\times n}} \frac{p-2}{p} \qquad (2.2.2)$$

This can be written in the form :

$$G(n) \sim \pi(n) \cdot \prod_{\substack{2\le p<\sqrt{n} \\ p/n}} (p-1) \cdot \prod_{\substack{3\le p<\sqrt{n} \\ p\times n}} (p-2) \Big/ \prod_{3\le p<\sqrt{n}} (p-1) \qquad (2.2.3)$$

Simplifying the quotient, and counting partitions into prime numbers q and $n-q$, with $q \le n/2$, we obtain the formula (2a).

Obviously, the formulas (2a) and

$$g'_{alt}(n) \sim \frac{1}{2} \cdot \frac{n}{\log(n)} \cdot \prod_{\substack{3< p<\sqrt{n} \\ p\times n}} \frac{p-2}{p-1} \quad \text{, and} \qquad (2b)$$

$$g''_{alt} \sim \frac{n}{4} \cdot e^{\gamma} \cdot \prod_{\substack{2\le p<\sqrt{n} \\ p/n}} \frac{p-1}{p} \cdot \prod_{\substack{3\le p<\sqrt{n} \\ p\times n}} \frac{p-2}{p} \qquad (2c)$$

are equivalent, as are the formulas (1.1, 1.2), and (2) :

$$\frac{g_{alt}}{g_{HL}} = \left[\frac{\pi(n)}{n/\log(n)}\right] \cdot \left[\frac{1}{1/\log(n)} \cdot \frac{e^{\gamma}}{2} \cdot \prod_{2\le p<\sqrt{n}} \frac{(p-1)}{p}\right] \cdot \left[\prod_{\sqrt{n}<p\le\infty} \frac{(p-1)^2}{p\cdot(p-2)}\right] \sim 1,$$

each of the three factors in square brackets tending towards unity.

Results

A preliminary numerical verification did show, that the results calculated with (2a) are not necessarily more accurate than those computed with (2b) or (2c), which are both about equally precise.

On **Table 1** and **Fig. 2**, the values for $g_{alt}(n)$, which have been calculated with the formula (2c) are compared with $g(n)$, as well with those calculated with the « corrected » Hardy-Littlewood formula*)

$$g_{SW}(n) \sim C_{HL} \cdot \frac{n}{(\log^2 n - 2 \cdot \log n)} \cdot \prod_{\substack{3 \le p \le \sqrt{n} \\ p / n}} \frac{p-1}{p-2} \tag{1.3}$$

Discussion

For its simplicity, the formula $g_{alt}(n)$ adequately describes the behaviour of $g(n)$, though it does not quite attain the degree of precision granted by $g_{SW}(n)$. The latter seems to be a remarkably accurate formula, indeed. After testing several formulas up to $n \le 5 \cdot 10^8$, Richstein $[3]$ finds it "a bit surprising" , that $g_{SW}(n)$ yields better results, on the whole, than other, more elaborate ones (with the exception of Selmer's formulas).

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*) In their paper $[1]$, Hardy and Littlewood stated, that, in the formula (1.1), the term $\log^2(n)$ of the denominator *is certainly in error wrong to an order* $\log(n)$ and, as had been suggested by Shah and Wilson, *must be replaced by* $\log^2(n) - 2 \cdot \log(n)$, a substitution, which is asymptotically irrelevant, but *essential for the purpose of verification within the limits of calculation*.

References

[1] G.H.Hardy, J.E. Littlewood, *Some problems of 'partitio numerorum'; III: On the expression of a number as a sum of primes*, Acta Math. **44** (1922), 32 – 39

[2] Wang Yuan (Ed.), *Goldbach Conjecture*, 1984, World Scientific Publishing, Singapore, ISBN 9971-966-08-5, and ISBN 9971-966-09-3 pbk (a collection of selected papers, containing a photographic reproduction of [1])

[3] Richstein J. ; Bosma Wieb (Editor*); Computing the number of Goldbach partitions up to 5.10*^{8}, Conference : Algorithmic number theory. International symposium, 4, Leiden, NLD, 2000-07-02 , ISSN : 0302-9743 ISBN : 3-540-67695-3

Table 1 Calculated results $g_{alt}(n)$ and $g_{SW}(n)$ compared with $g(n)$

n			*g(n)*	$g_{alt}(n)$	$g_{SW}(n)$	$g_{alt}(n)$	$g_{SW}(n)$	$(g_{alt}-g_{SW})/g_{SW}$
						Relative	Error	
2*3*5	=	30	3	4	11	33,33%	266,67%	-63,64%
2*3*5*7	=	210	19	15	25	-21,05%	31,58%	-40,00%
2*3*5*7*11	=	2310	114	93	122	-18,42%	7,02%	-23,77%
2*3*5*7*11*13	=	30030	905	788	898	-12,93%	-0,77%	-12,25%
2*3*5*7*11*13*17	=	510510	9493	8969	9521	-5,52%	0,29%	-5,80%
2*3*5*7*11*17*19*23*29	=	497668710*)	3977551*)	4006949	3970818	0,74%	-0,17%	0,91%
		1000	28	18	26	-35,71%	-7,14%	-30,77%
		10000	127	114	133	-10,24%	4,72%	-14,29%
		100000	810	731	804	-9,75%	-0,74%	-9,08%
		1000000	5402	5139	5392	-4,87%	-0,19%	-4,69%
		10000000	38807	37974	38681	-2,15%	-0,32%	-1,83%
		100000000	291400	290577	290999	-0,28%	-0,14%	-0,15%
2^{10}	=	1024	22	14	20	-36,36%	-9,09%	-30,00%
2^{11}	=	2048	25	24	32	-4,00%	28,00%	-25,00%
2^{12}	=	4096	53	42	51	-20,75%	-3,77%	-17,65%
2^{13}	=	8192	76	71	86	-6,58%	13,16%	-17,44%
2^{14}	=	16384	151	125	145	-17,22%	-3,97%	-13,79%
2^{15}	=	32768	244	217	248	-11,07%	1,64%	-12,50%
2^{16}	=	65536	435	388	429	-10,80%	-1,38%	-9,56%
2^{17}	=	131072	749	689	751	-8,01%	0,27%	-8,26%
2^{18}	=	262144	1314	1228	1324	-6,54%	0,76%	-7,25%
2^{19}	=	524288	2367	2220	2353	-6,21%	-0,59%	-5,65%
2^{20}	=	1048576	3957	4010	4209	1,34%	6,37%	-4,73%
2^{21}	=	2097152		7302	7575			-3,60%
2^{22}	=	4194304		13305	13705			-2,92%
2^{23}	=	8388608		24340	24914			-2,31%
2^{24}	=	16777216		44750	45491			-1,63%
2^{25}	=	33554432		82614	83393			-0,93%
2^{26}	=	67108864		152885	153433			-0,36%
2^{27}	=	134217728		283717	283247			0,17%
2^{28}	=	268435456		527313	524511			0,53%
2^{29}	=	536870912		983427	974065			0,96%

*)For $6 \le n \le 5 \cdot 10^8$, the maximal value assumed by $g(n)$ (Richstein[3])

Fig. 2a Calculated results $g_{alt}(n)$ compared with $g(n)$

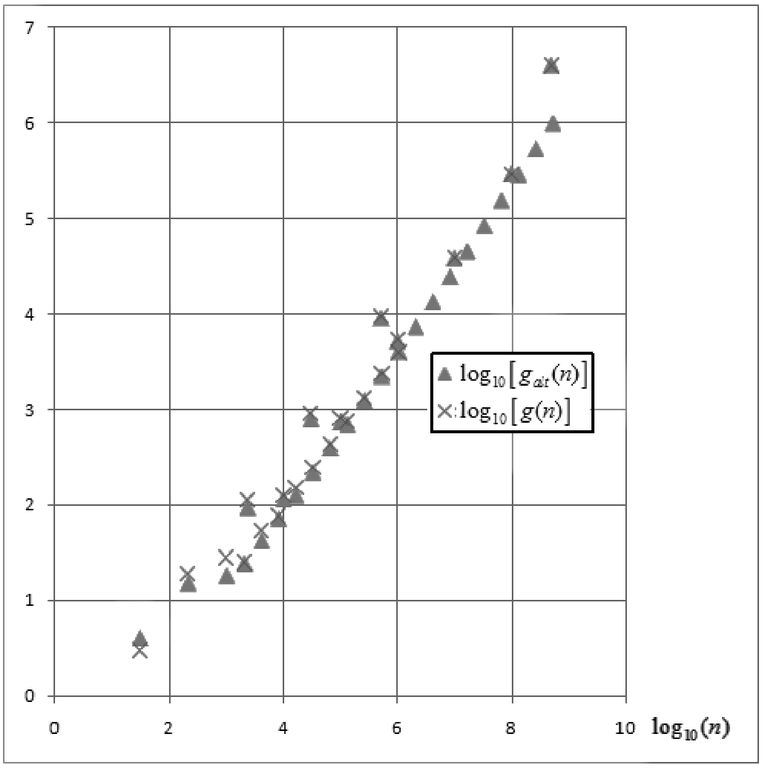

Fig. 2b Calculated results $g_{alt}(n)$ compared with $g_{SW}(n)$