APPENDIX

February 2011

to the article Estimating the Number of Goldbach Partitions

As Greg Bernhardt has signalled to me, the formula, which I indicated for $g_{alt}(n)$ is asymptotically not correct. Now, the following should be exact:

1. Asymptotic behaviour

To show, that the formulas

$$g_{ALT}(n) = \frac{e^{\gamma}}{4} \cdot \frac{n}{\log(n)} \cdot \prod_{\substack{3 \le p < \sqrt{n} \\ n \not\equiv 0 (\mathrm{mod}\, p)}} \frac{p-2}{p-1} \text{, and} \tag{A1.1}$$

$$g_{HL}(n) = C_2 \cdot \frac{n}{\log^2(n)} \cdot \prod_{\substack{3 \le p < \sqrt{n} \\ n \equiv 0 (\mathrm{mod}\, p)}} \frac{p-1}{p-2} \text{, with } C_2 = \prod_{3 \le p < \infty} \frac{p \cdot (p-2)}{(p-1)^2} \tag{A1.2}$$

are asymptotically equivalent, we form the quotient

$$\frac{g_{ALT}(n)}{g_{HL}(n)} = \frac{e^{\gamma}}{4} \cdot \frac{1}{C_2} \cdot \log(n) \cdot \prod_{3 \le p < \sqrt{n}} \frac{p-2}{p-1} \tag{A1.3}$$

(which, by the way, does not depend on the factorization of n), and use Merten's theorem to obtain:

$$\frac{g_{ALT}(n)}{g_{HL}(n)} \sim \frac{1}{C_2} \cdot \prod_{3 \le p < \sqrt{n}} \frac{p \cdot (p-2)}{(p-1)^2} <\sim 1 . \tag{A1.4}$$

Fig. 1A (see section 3 of this paper, Calculated results) illustrates, over a wide range of powers of 10, the behaviour of the difference $1 - g_{ALT}(n) / g_{HL}(n)$.

2. The alternative Formula can be supported by heuristic reasonment as follows:

Numbers q $(1 \le q < n)$ which lend themselves to a Goldbach Partition (let's name them "Goldbach numbers") must satisfy the two conditions:

$q \not\equiv 0 (\mathrm{mod}\, p)$ and $q \not\equiv n (\mathrm{mod}\, p)$ for all prime numbers p ($2 \le p < \sqrt{n}$) (A2.1)

Among the prime numbers p ($2 \le p < \sqrt{n}$), let's denote with p' those, which are prime factors of n, and with p'' the rest:

$n \equiv 0 (\mathrm{mod}\, p')$, and $n \not\equiv 0 (\mathrm{mod}\, p'')$. (A2.2)

Now, the conditions can be stated as follows:

- with regard to p': $q' \not\equiv 0(\bmod p')$, (which entails $q' \not\equiv n(\bmod p')$, and vice versa) (A2.3)

- with regard to p'': $q'' \not\equiv 0(\bmod p'')$ *and* $q'' \not\equiv n(\bmod p'')$ (A2.4)

Let's denote with Q' and Q'' the sets consisting of the numbers q', q'' respectively . The intersection $Q' \cap Q''$ consists of the "Goldbach numbers" $q > n$ (including the numbers 1 and $(n-1)$, if the latter happens to be a prime number, but excluding all prime numbers $p < \sqrt{n}$ and their counterpart ($n-p$). For asymptotic considerations, these exceptions are obviously irrelevant).

Asymptotically, we estimate the probability for a number q $(1 \le q < n)$ belonging to one of the sets Q'_n , Q''_n to be equal to

$$P'_n = \frac{e^\gamma}{2} \cdot \prod_{2 \le p' < \sqrt{n}} \frac{p'-1}{p'}, \quad P''_n = \frac{e^\gamma}{2} \cdot \prod_{3 \le p'' < \sqrt{n}} \frac{p''-2}{p''} \text{, respectively,} \tag{A2.5}$$

and, for it belonging to the intersection $Q'_n \cap Q''_n$, to

$$P'_n \cdot P''_n = \left(\frac{e^\gamma}{2}\right)^2 \cdot \prod_{2 \le p' < \sqrt{n}} \frac{p'-1}{p'} \cdot \prod_{3 \le p'' < \sqrt{n}} \frac{p''-2}{p''}. \tag{A2.6}$$

Multiplying this by $n/2$ (counting partitions q and $n-q$, with $q \le n/2$), we obtain, as an estimate for the count of partitions:

$$g^*_{ALT}(n) = \frac{n}{2} \cdot \left(\frac{e^\gamma}{2}\right)^2 \cdot \prod_{2 \le p' < \sqrt{n}} \frac{p'-1}{p'} \cdot \prod_{3 \le p'' < \sqrt{n}} \frac{p''-2}{p''} \tag{A2.7}$$

Dividing this by $\frac{1}{2} \cdot e^\gamma \cdot \prod_{2 \le p < \sqrt{n}} \frac{p-1}{p} \cdot \log(n) \sim 1$ (Merten's theorem), we get (A2.8)

$$g^*_{ALT}(n) \sim g_{ALT}(n) \text{ with } g_{ALT}(n) = \frac{e^\gamma}{4} \cdot \frac{n}{\log(n)} \cdot \prod_{3 \le p'' < \sqrt{n}} \frac{p''-2}{p''-1}. \tag{A2.9}$$

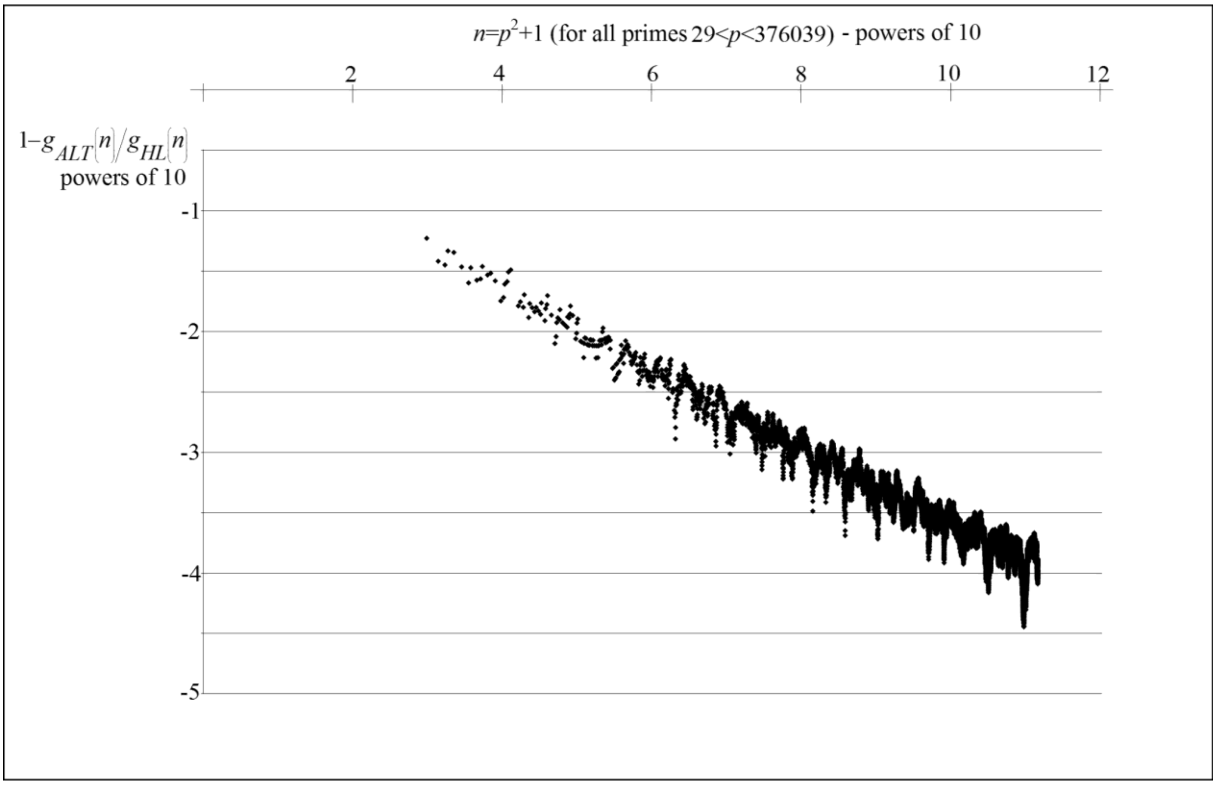
Let's point out, finally, in which respects our deduction remains hypothetical:

- It is probabilistic. And in A2.6 we have assumed , that the probabilities P'_n and P''_n are independent.

- The Mertens theorem implies an asymptotic estimation of the density $1/\log(n)$ of the prime numbers inferior to n . For this, it requires the product being extended over *all* prime numbers p ($2 \le p < \sqrt{n}$). In order to arrive (in A2.5) at density estimations for the subsets Q'_n , Q''_n, we have taken the liberty of "generalizing" the theorem, extending the product over the prime numbers p', p'', respectively.

3. Calculated results

Figure 1A Plot of $1-g_{ALT}(n)/g_{HL}(n)$



In Table 1A , values calculated with the formulas

$$g_{ALT}(n)=\frac{e^{\gamma}}{4}\cdot\frac{n}{\log(n)}\cdot\prod_{\substack{3\le p<\sqrt{n}\\ n\not\equiv 0(\bmod p)}}\frac{p-2}{p-1}, \qquad \text{(A1.1)}$$

$$g^{\pi}_{ALT}(n)=\frac{e^{\gamma}}{4}\cdot\pi(n)\cdot\prod_{\substack{3\le p<\sqrt{n}\\ n\not\equiv 0(\bmod p)}}\frac{p-2}{p-1}, \text{ and}$$

$$g_{HL}(n)=C_2\cdot\frac{n}{\log^2(n)}\cdot\prod_{\substack{3\le p<\sqrt{n}\\ n\equiv 0(\bmod p)}}\frac{p-1}{p-2}, \text{ with } C_2=\prod_{3\le p<\infty}\frac{p\cdot(p-2)}{(p-1)^2} \qquad \text{(A1.2)}$$

are compared against the exact values of $g(n)$.

Table 1A

- n	$\pi(n)$	$g(n)$	$g_{HL}(n)$	$g_{ALT}(n)$	$g^{\pi}_{ALT}(n)$
2*3*5	10	3	5	4	4
2*3*5*7	46	19	16	14	17
2*3*5*7*11	343	114	90	87	100
2*3*5*7*11*13	3248	905	723	713	795
2*3*5*7*11*13*17	42331	9493	8072	8030	8751
2*3*5*7*11*17*19*23*29	26239628	3977551	3574241	3571293	3770726
10^3	168	28	18	17	20
10^4	1229	127	104	103	116
10^5	9592	810	664	657	726
10^6	78498	5402	4612	4594	4983
10^7	664579	38807	33881	33850	36259
10^8	5761455	291400	259405	259088	274970
2^{10}	172	22	14	13	15
2^{11}	309	25	23	22	26
2^{12}	564	53	39	38	44
2^{13}	1028	76	67	65	74
2^{14}	1900	151	115	113	127
2^{15}	3512	244	200	197	219
2^{16}	6542	435	352	349	386
2^{17}	12251	749	623	619	681
2^{18}	23000	1314	1112	1103	1207
2^{19}	43390	2367	1996	1986	2165
2^{29}	28192750		877149	876465	925178

$g(n) > g^{\pi}_{ALT}(n) > g_{HL}(n) > g_{ALT}(n)$ (for $g(n) > 25$)