The model assembly procedure for a transient heat transfer problem is exactly the same as for a steady-state problem, with the notable exception that we must also assemble a global capacitance matrix. The rules are the same. Element nodes are assigned to global nodes and the element capacitance matrix terms are added to the appropriate global positions in the global capacitance matrix, as with the conductance matrix terms. Hence, on system assembly, we obtain the global equations

$$[C]\{\dot{T}\} + [K]\{T\} = \{F_Q\} + \{F_g\} \tag{7.116}$$

where we must recall that the gradient force vector $\{F_g\}$ is composed of either (1) unknown heat flux values to be determined (unknown reactions) or (2) convection terms to be equilibrated with the flux at a boundary node.

7.8.1 Finite Difference Methods for the Transient Response: Initial Conditions

The finite element discretization procedure has reduced the one-dimensional transient heat transfer problem to algebraic terms in the spatial variable via the interpolation functions. Yet Equation 7.116 represents a set of ordinary, coupled, first-order differential equations in time. Consequently, as opposed to the steady-state case, there is not *a* solution but multiple solutions as the system responds to time-dependent conditions. The boundary conditions for a transient problem are of the three types discussed for the steady-state case: specified nodal temperatures, specified heat flux, or convection conditions. However, note that the boundary conditions may also be time dependent. For example, a specified nodal temperature could increase linearly with time to some specified final value. In addition, an internal heat generation source Q may also vary with time.

A commonly used approach to obtaining solutions for ordinary differential equations of the form of Equation 7.116 is the *finite difference method*. As discussed briefly in Chapter 1, the finite difference method is based on approximating derivatives of a function as incremental changes in the value of the function corresponding to finite changes in the value of the independent variable. Recall that the first derivative of a function $f(t)$ is defined by

$$\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{7.117}$$

Instead of requiring Δt to approach zero, we obtain an approximation to the value of the derivative by using a small, nonzero value of Δt to obtain

$$\dot{f} \cong \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{7.118}$$

and the selected value of Δt is known as the *time step*.

To apply the procedure to transient heat transfer, we approximate the time derivative of the nodal temperature matrix as

$$\{\dot{T}\} \cong \frac{\{T(t + \Delta t)\} - \{T(t)\}}{\Delta t} \tag{7.119}$$

Substituting, Equation 7.116 becomes

$$[C]\frac{\{T(t+\Delta t)\}-\{T(t)\}}{\Delta t}+[K]\{T(t)\}=\{F_Q(t)\}+\{F_g(t)\} \qquad (7.120)$$

Note that, if the nodal temperatures are known at time t and the forcing functions are evaluated at time t, Equation 7.120 can be solved, algebraically, for the nodal temperatures at time $t+\Delta t$. Denoting the time at the ith time step as $t_i = i(\Delta t)$, $i = 0, 1, 2, \ldots,$ we obtain

$$[C]\{T(t_{i+1})\} = [C]\{T(t_i)\} - [K]\{T(t_i)\}\,\Delta t + \{F_Q(t_i)\}\Delta t + \{F_g(t_i)\}\Delta t \qquad (7.121)$$

as the system of algebraic equations that can be solved for $\{T(t_{i+1})\}$. Formally, the solution is obtained by multiplying Equation 7.121 by the inverse of the capacitance matrix. For large matrices common to finite element models, inverting the matrix is very inefficient, so other techniques such as Gaussian elimination are more often used. Note, however, that the system of algebraic equations given by Equation 7.121 must be solved only once to obtain an *explicit* solution for the nodal temperatures at time t_{i+1}.

The method just described is known as a *forward difference scheme* (also known as *Euler's method*) and Equation 7.121 is a *two-point recurrence relation*. If the state of the system (nodal temperatures and forcing functions) is known at one point in time, Equation 7.121 gives the state at the next point in time. Solving the system sequentially at increasing values of the independent variable is often referred to as *marching* in time. To begin the solution procedure, the state of the system must be known at $t = 0$. Therefore, the *initial conditions* must be specified in addition to the applicable boundary conditions. Recall that the general solution to an ordinary, first-order differential equation contains one constant of integration. As we have one such equation corresponding to each nodal temperature, the value of each nodal temperature must be specified at time zero. If the initial conditions are so known, the recurrence relation can be used to compute succeeding nodal temperatures. Prior to discussing other schemes and the ramifications of time step selection, the following simple example is presented.

EXAMPLE 7.11

Figure 7.20a shows a cylindrical rod having diameter of 12 mm and length of 100 mm. The pin is of a material having thermal conductivity 230 W/(m-°C), specific heat 900 J/(kg-°C), and density 2700 kg/m^3. The right-hand end of the rod is held in contact with a medium at a constant temperature of 30°C. At time zero, the entire rod is at a temperature of 30°C when a heat source is applied to the left end, bringing the temperature of the left end immediately to 80°C and maintaining that temperature indefinitely. Using the forward difference method and four two-node elements, determine both transient and steady-state temperature distributions in the rod. No internal heat is generated.

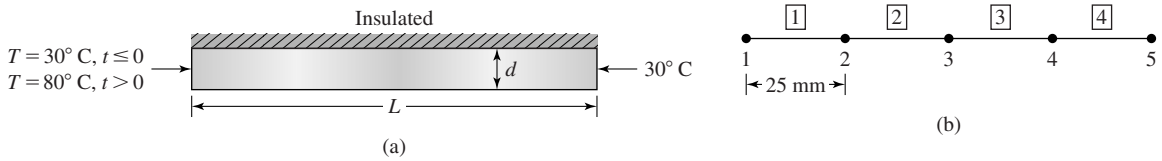

Figure 7.20
(a) Cylindrical rod of Example 7.11. (b) Node and element numbers.

■ Solution

In the solution for this example, we set up the general procedure then present the results for one solution using one time step for the transient portion. The node numbers and element numbers are as shown in Figure 7.20b. Since the length and area of each element are the same, we compute the element capacitance matrix as

$$\left[C^{(e)}\right] = \frac{c\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{900(2700)\dfrac{\pi}{4}(0.012)^2(0.025)}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2.2902 & 1.1451 \\ 1.1451 & 2.2902 \end{bmatrix} \text{J/°C}$$

where we have implicitly performed the integrations indicated in Equation 7.115 and leave the details as an end-of-chapter exercise. Similarly, the element conductance matrix is

$$\left[k^{(e)}\right] = \frac{kA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{200\dfrac{\pi}{4}(0.012)^2}{0.025}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.9408 & -0.9408 \\ -0.9408 & 0.9408 \end{bmatrix} \text{W/°C}$$

For the one-dimensional case with uniform geometry and material properties, the system assembly is straightforward and results in the global matrices

$$[C] = \begin{bmatrix} 2.2902 & 1.1451 & 0 & 0 & 0 \\ 1.1451 & 4.5804 & 1.1451 & 0 & 0 \\ 0 & 1.1451 & 4.5804 & 1.1451 & 0 \\ 0 & 0 & 1.1451 & 4.5804 & 1.1451 \\ 0 & 0 & 0 & 1.1451 & 2.2902 \end{bmatrix}$$

$$[K] = \begin{bmatrix} 0.9408 & -0.9408 & 0 & 0 & 0 \\ -0.9408 & 1.8816 & -0.9408 & 0 & 0 \\ 0 & -0.9408 & 1.8816 & -0.9408 & 0 \\ 0 & 0 & -0.9408 & 1.8816 & -0.9408 \\ 0 & 0 & 0 & -0.9408 & 0.9408 \end{bmatrix}$$

As no internal heat is generated $\{F_Q\} = 0$ and, as we have specified boundary temperatures, the flux forcing term is an unknown. Note that, in the transient case, the flux terms

at the boundaries (the "reactions") are time dependent and can be computed at each time step, as will be explained. Hence, the gradient "force vector" is

$$\{F_g\} = \begin{Bmatrix} q_1 A \\ 0 \\ 0 \\ 0 \\ -q_5 A \end{Bmatrix}$$

Having taken care of the boundary conditions, we now consider the initial conditions and examine the totality of the conditions on the solution procedure. It should be clear that, since we have the temperature of two nodes specified, the desired solution should provide the temperatures of the other three nodes and, therefore, should be a 3×3 system. The reduction to the 3×3 system is accomplished via the following observations:

1. If $T_1 = 80^\circ\text{C} = \text{constant}$, then $\dot{T}_1 = 0$.

2. If $T_5 = 30^\circ\text{C} = \text{constant}$, then $\dot{T}_5 = 0$.

The equations can be modified accordingly. In this example, the general equations become

$$[C]\begin{Bmatrix} 0 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ 0 \end{Bmatrix} + [K]\begin{Bmatrix} 80 \\ T_2 \\ T_3 \\ T_4 \\ 30 \end{Bmatrix} = \begin{Bmatrix} q_1 A \\ 0 \\ 0 \\ 0 \\ -q_5 A \end{Bmatrix}$$

Consequently, the first and fifth equations become

$$1.1451\,\dot{T}_2 + 0.9408(80) - 0.9408\,T_2 = q_1 A$$

$$1.1451\,\dot{T}_4 - 0.9408\,T_4 + 0.9408(30) = -q_5 A$$

respectively. The three remaining equations are then written as

$$\begin{bmatrix} 4.5804 & 1.1451 & 0 \\ 1.1451 & 4.5804 & 1.1451 \\ 0 & 1.1451 & 4.5804 \end{bmatrix}\begin{Bmatrix} \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \end{Bmatrix} + \begin{bmatrix} 1.8816 & -0.9408 & 0 \\ -0.9408 & 1.8816 & -0.9408 \\ 0 & -0.9408 & 1.8816 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

$$= \begin{Bmatrix} 75.264 \\ 0 \\ 28.224 \end{Bmatrix}$$

For this example, the capacitance matrix is inverted (using a spreadsheet program) to obtain

$$[C]^{-1} = \begin{bmatrix} 0.2339 & -0.0624 & 0.0156 \\ -0.0624 & 0.2495 & -0.0624 \\ 0.0156 & -0.0624 & 0.2339 \end{bmatrix}$$

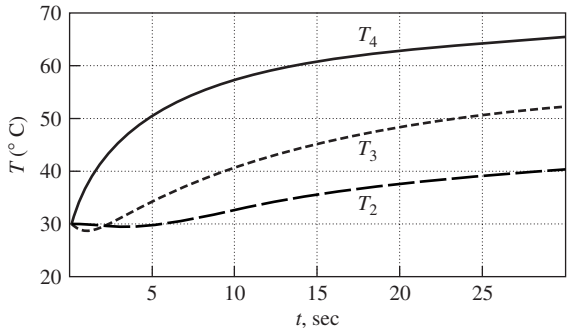

Figure 7.21 Time histories of the nodal temperatures.

where $[C]$ now represents the reduced 3×3 capacitance matrix. Utilizing Equation 7.121 and multiplying by $[C]^{-1}$ yields

$$\begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix}_{i+1} = \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix}_i - \begin{bmatrix} 0.4988 & -0.3521 & 0.0880 \\ -0.3521 & 0.5869 & -0.3521 \\ 0.0880 & -0.3521 & 0.4988 \end{bmatrix} \begin{Bmatrix} T_2 \\ T_3 \\ T_4 \end{Bmatrix}_i \Delta t + \begin{Bmatrix} 18.0456 \\ -6.4558 \\ 7.7762 \end{Bmatrix} \Delta t$$

as the two-point recurrence relation.

Owing to the small matrix involved, the recurrence relation was programmed into a standard spreadsheet program using time step $\Delta t = 0.1$ sec. Calculations for nodal temperatures T_2, T_3, and T_4 are carried out until a steady state is reached. Time histories of each of the nodal temperature are shown in Figure 7.21. The figure shows that steady-state conditions $T_2 = 67.5^\circ$C, $T_3 = 55^\circ$C, and $T_4 = 42.5^\circ$C are attained in about 30 sec. Interestingly, the results also show that the temperatures of nodes 3 and 4 initially decrease. Such phenomena are physically unacceptable and associated with use of a consistent capacitance matrix, as is discussed in Chapter 10.

7.8.2 Central Difference and Backward Difference Methods

The forward difference method discussed previously and used in Example 7.11 is but one of three commonly used finite difference methods. The others are the backward difference method and the central difference method. Each of these is discussed in turn and a single two-point recurrence relation is developed incorporating the three methods.

In the *backward difference method,* the finite approximation to the first derivative at time t is expressed as

$$\dot{T}(t) \cong \frac{T(t) - T(t - \Delta t)}{\Delta t} \tag{7.122}$$

so that we, in effect, look back in time to approximate the derivative during the previous time step. Substituting this relation into Equation 7.116 gives

$$[C]\frac{\{T(t)\} - \{T(t - \Delta t)\}}{\Delta t} + [K]\{T(t)\} = \{F_Q(t)\} + \{F_g(t)\} \tag{7.123}$$

In this method, we evaluate the nodal temperatures at time t based on the state of the system at time $t - \Delta t$, so we introduce the notation $t = t_i$, $t_{i-1} = t - \Delta t$, $i = 1, 2, 3, \ldots$. Using the described notation and rearranging, Equation 7.123 becomes

$$([C] + [K]\Delta t)\{T(t_i)\} = [C]\{T(t_{i-1})\} + F_Q(t_i)\Delta t + F_g(t_i)\Delta t \qquad i = 1, 2, 3, \ldots \tag{7.124}$$

If the nodal temperatures are known at time t_{i-1}, Equation 7.124 can be solved for the nodal temperatures at the next time step (it is assumed that the forcing functions on the right-hand side are known and can be determined at t_i). Noting that the time index is relative, Equation 7.125 can also be expressed as

$$\begin{aligned}&([C] + [K]\Delta t)\{T(t_{i+1})\}\\ &\quad = [C]\{T(t_i)\} + F_Q(t_i)\Delta t + F_g(t_i)\Delta t \qquad i = 0, 1, 2, \ldots\end{aligned} \tag{7.125}$$

If we compare Equation 7.125 with Equation 7.121, we find that the major difference lies in the treatment of the conductance matrix. In the latter case, the effects of conductance are, in effect, updated during the time step. In the case of the forward difference method, Equation 7.121, the conductance effects are held constant at the previous time step. We also observe that Equation 7.125 cannot be solved at each time step by "simply" inverting the capacitance matrix. The coefficient matrix on the left-hand side changes at each time step; therefore, more efficient methods are generally used to solve Equation 7.125.

Another approach to approximation of the first derivative is the central difference method. As the name implies, the method is a compromise of sorts between forward and backward difference methods. In a central difference scheme, the dependent variable and all forcing functions are evaluated at the center (midpoint) of the time step. In other words, average values are used. In the context of transient heat transfer, the time derivative of temperature is still as approximated by Equation 7.119 but the other terms in Equation 7.120 are evaluated at the midpoint of the time step. Using this approach, Equation 7.120 becomes

$$\begin{aligned}&[C]\frac{\{T(t + \Delta t)\} - \{T(t)\}}{\Delta t} + [K]\left\{\frac{T(t + \Delta t) + T(t)}{2}\right\}\\ &\quad = \left\{\frac{F_Q(t + \Delta t) + F_Q(t)}{2}\right\} + \left\{\frac{F_g(t + \Delta t) + F_g(t)}{2}\right\}\end{aligned} \tag{7.126}$$

The forcing functions on the right-hand side of Equation 7.126 are either known functions and can be evaluated or "reactions," which are subsequently computed

via the constraint equations. The left-hand side of Equation 7.126 is now, however, quite different, in that the unknowns at each step $T_i(t + \Delta t)$ appear in both capacitance and conductance terms. Multiplying by Δt and rearranging Equation 7.126, we obtain

$$\left([C] + [K]\frac{\Delta t}{2}\right)\{T(t + \Delta t)\} = \left([C] - [K]\frac{\Delta t}{2}\right)\{T(t)\} + \left\{\frac{F_Q(t + \Delta t) + F_Q(t)}{2}\right\} + \left\{\frac{F_g(t + \Delta t) + F_g(t)}{2}\right\} \tag{7.127}$$

Equation 7.127 can be solved for the unknown nodal temperatures at time $t + \Delta t$ and the "marching" solution can progress in time until a steady state is reached. The central difference methods is, in general, more accurate than the forward or backward difference method, in that it does not give preference to either temperatures at t or $t + \Delta t$ but, rather, gives equal credence to both.

In finite difference methods, the key parameter governing solution accuracy is the selected time step Δt. In a fashion similar to the finite element method, in which the smaller the elements are, physically, the better is the solution, the finite difference method converges more rapidly to the true solution as the time step is decreased. These ideas are amplified in Chapter 10, when we examine the dynamic behavior of structures.

7.9 CLOSING REMARKS

In Chapter 7, we expand the application of the finite element method into two- and three-dimensional, as well as axisymmetric, problems in heat transfer. While the majority of the chapter focuses on steady-state conditions, we also present the finite difference methods commonly used to examine transient effects. The basis of our approach is the Galerkin finite element method, and this text stays with that procedure, as it is so general in application. As we proceed into applications in fluid mechanics, solid mechanics, and structural dynamics in the following chapters, the Galerkin method is the basis for the development of many of the finite element models.

REFERENCES

1. Huebner, K. H., and E. A. Thornton. *The Finite Element Method for Engineers,* 2nd ed. New York: John Wiley and Sons, 1982.
2. Incropera, F. P., and D. P. DeWitt. *Introduction to Heat Transfer,* 3rd ed. New York: John Wiley and Sons, 1996.