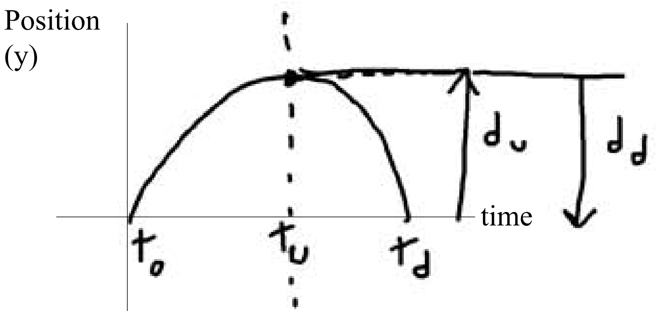
Proof that projectile motion under ideal conditions is "symmetrical"

In this proof, I attempt to show that, for an object undergoing projectile motion under idealized conditions (no air resistance, etc.), the magnitude of the object's initial velocity is equal to the magnitude of the object's velocity when it has returned to its original y-position. I also attempt to show that the time taken for the object to reach its maximum y-position is the same as the time taken to return from the maximum y-position to the initial y-position.

Consider the trajectory followed by an object which has been launched into the air, and is undergoing projectile motion under idealized conditions. The graph of such an object's y-position against time would look something like this:



Now it will help to define some variables which will help us discuss the left and right halves of the motion in this graph.

t_0 = Start time of the motion

t_u = How long it takes for the object to achieve its maximum y-position

t_d = How long it takes for the object to return from its maximum y-position to its original y-position

d_u = The change in position from the initial y-value to the maximum y-value

d_d = The change in position from the maximum y-value back to the initial y-value

V_{ou} = The object's initial velocity in the y-direction

V_u = The object's velocity in the y-direction at time t_u, which is obviously 0m/s

V_{od} = Another name for V_u, this will make it easier to think about parts of the proof.

V_d = The object's velocity in the y-direction at time t_d

So, from the definitions provided above, we can conclude that $d_u = -d_d$, $V_u = 0\,m/s$, $V_{od} = 0\,m/s$
In all cases.

Now let V_{ou} be any positive real number velocity value.

Consider the kinematic equation, $d = \frac{1}{2}(V_0 + V)t$

Applying this equation to the first half of the motion, we have

$$d_u = \frac{1}{2}(V_{ou} + \cancel{V_u}^{0})\,t_u$$

$$\rightarrow t_u = \frac{2d_u}{V_{ou}} \quad * V_u = 0\,m/s \text{ by definition}$$

Applying this same kinematic equation to the second half of the motion, we have

$$d_d = \frac{1}{2}(\cancel{V_{od}}^{0} + V_d)\,t_d$$

$$\rightarrow t_d = \frac{2d_d}{V_d} \quad * V_{od} = 0\,m/s \text{ by definition}$$

Now consider another kinematic equation, $V = V_0 + at$

Applying this to the two equations just obtained above, we have

$$t_u = \frac{2d_u}{V_{0u}}$$

$$t_u = \frac{2d_u}{\cancel{V_u}^{0} - at_u}$$ *Because $V_{0u} = V_u - at_u$ from the kinematic equation

$$t_u = \frac{-2d_u}{at_u}$$ * $V_u = 0\,m/s$ by definition

$$t_u^2 = -\frac{2}{a}d_u$$

$$t_d = \frac{2d_d}{V_d}$$

$$t_d = \frac{2d_d}{\cancel{V_{0d}}^{0} + at_d}$$ * Because $V_d = V_{0d} + at_d$ from the kinematic equation

$$t_d = \frac{2d_d}{at_d}$$ * Because $V_{0d} = 0\,m/s$ by definition

$$t_d^2 = \frac{2}{a}d_d$$

$$t_d^2 = \frac{2}{a}(-d_u)$$ * we know that $d_d = -d_u$

$$t_d^2 = -\frac{2}{a}d_u$$

* observe that $t_u^2 = -\frac{2}{a}d_u = t_d^2$, so $t_u = t_d$

So, we have seen that in the generic case, when the distance travelled up is the same as the distance travelled back down, the time taken to complete either half of the full trip is the same, no matter what the objects initial velocity was when it was launched off the ground.

Using this result, let's again consider the kinematic equation $V = V_0 + at$ applied to both halves of the motion:

$$V_u = V_{0u} + at_u$$

$$\rightarrow t_u = \frac{\cancel{V_u}^{0} - V_{0u}}{a}$$

$$\rightarrow t_u = -\frac{V_{0u}}{a}$$ * $V_u = 0\,m/s$ by definition

$$V_d = V_{0d} + at_d$$

$$\rightarrow t_d = \frac{V_d - \cancel{V_{0d}}^{0}}{a}$$

$$\rightarrow t_d = \frac{V_d}{a}$$ * $V_{0d} = 0\,m/s$ by definition

Since we know that $t_u = t_d$, we have $-\frac{V_{0u}}{a} = \frac{V_d}{a}$

$$\therefore V_d = -V_{0u}$$

Thus, the velocity of the object when it returns to its initial y-position must be equivalent to the initial velocity with which it was launched.

Now, A further question is this -- supposedly, for any y-position, the object is moving at the same velocity (of course, opposite direction), when it crosses this position on either half of its trip, up or down. In my proof, I decided to only show that the initial and final velocities are the same, because it seemed easier. Can you think of a way to show that the above statement holds?