THE NONEXISTNECE OF {4,5,7}

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Let the number α be defined as:

$$\alpha \mid (\prod_{k=1}^{\alpha} k) + 1 = x^2$$

Where x is a natural number.

We could alternatively express this as:

$$\alpha \mid \alpha! + 1 = x^2$$

Which is Brocard's problem. The known values of α are 4, 5 and 7. The above equation can be rewritten as, due to Euler's reflection formula:

$$\frac{\alpha\pi}{\Gamma(1-\alpha)\sin(\alpha\pi)} + 1 = x^2$$

Also,

$$\Gamma(1-\alpha) = \frac{\pi}{\Gamma(\alpha)\sin(\alpha\pi)}$$

Thus,

$$\alpha \sin(\alpha\pi)\Gamma(\alpha) + 1 = x^2$$

Euler's identity is:

$$i \sin x + \cos x = e^{ix}$$

From the 2 equations above, we see that:

$$\sin(\alpha\pi) = \frac{e^{i\pi\alpha} - \cos(\alpha\pi)}{i}$$

$$\alpha e^{i\pi\alpha} - \cos(\alpha\pi) = ix^2$$

Since $\cos(\alpha\pi) = \pm 1$ for all integer α's,

$$\alpha e^{i\pi\alpha} \mp 1 = ix^2$$

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$\alpha e^{i\pi\alpha} = ix^2 \mp 1$

$\alpha e^{i\pi\alpha} = ix^2 \pm 1$

$\alpha (e^{\pi i})^{\alpha} = ix^2 \pm 1$

$\alpha (-1)^{\alpha} = ix^2 \pm 1$

As

$(-1)^{\alpha} = 1 | \alpha \text{ even}$
$(-1)^{\alpha} = -1 | \alpha \text{ odd}$

$\alpha \pm 1 = ix^2 \pm 1$

Four possibilities

I. $\alpha + 1 = ix^2 + 1$

II. $\alpha - 1 = ix^2 - 1$

III. $\alpha + 1 = ix^2 - 1$

IV. $\alpha - 1 = ix^2 + 1$

I.

$\alpha = ix^2$

Which makes alpha imaginary.

II.

$\alpha = ix^2$

Which makes alpha imaginary.

III.

$\alpha = ix^2 - 2$

Which makes alpha complex (it is impossible that it is real as then x would be 0).

IV.

$\alpha = ix^2 + 2$

Which makes alpha complex and definitely not real due to the same argument as in Case III.

Conclusion

This leaves no room for real numbers. {4,5,7} are real and are really examples of alpha. This means 4, 5 and 7 do not exist.