and combinations of these, it is easy to see that all matrices in SO(3) are rotations, since reflections have determinant -1.

Example 3.33

A very interesting space arises by considering the set of all straight lines through the origin in $\mathbf{R}^3$, which we write $\mathbf{R}\mathrm{P}^2$ for. This is called the **real projective plane**. It is not a subset of $\mathbf{R}^3$, since the elements of $\mathbf{R}\mathrm{P}^2$ are not points in $\mathbf{R}^3$ but subsets of $\mathbf{R}^3$. So we cannot use the subspace topology to make it into a topological space.

Instead, we could topologize it as follows. If we take a subset of $\mathbf{R}\mathrm{P}^2$, i.e., a collection of lines in $\mathbf{R}^3$, then we can take the union of these lines to get a subset of $\mathbf{R}^3$. We could then define the subset of $\mathbf{R}\mathrm{P}^2$ to be open if the corresponding subset of $\mathbf{R}^3$ is open, i.e., $S \subset \mathbf{R}\mathrm{P}^2$ is open if $\bigcup_{l \in S} l \subset \mathbf{R}^3$ is open. Unfortunately, this gives the indiscrete topology on $\mathbf{R}\mathrm{P}^2$, since the union will contain 0, unless S is empty, but will not contain an open ball around 0 unless $S = \mathbf{R}\mathrm{P}^2$.

To avoid this problem, we omit 0, and we say that a subset $S \subset \mathbf{R}\mathrm{P}^2$ is open if the subset $\bigcup_{l \in S}(l - 0)$ of $\mathbf{R}^3 - \{0\}$ is open. The empty subset of $\mathbf{R}\mathrm{P}^2$ is then open, because the corresponding subset of $\mathbf{R}^3 - \{0\}$ is also the empty set. And the whole set $\mathbf{R}\mathrm{P}^2$ is open, because this corresponds to the whole set $\mathbf{R}^3 - \{0\}$.

Unions and intersections of subsets of $\mathbf{R}\mathrm{P}^2$ correspond to unions and intersections of subsets of $\mathbf{R}^3 - \{0\}$ so, because the open sets in $\mathbf{R}^3 - \{0\}$ form a topology, these open sets in $\mathbf{R}\mathrm{P}^2$ also form a topology.

Example 3.34

We can do something similar in $\mathbf{R}^n$ for any n: Define $\mathbf{R}\mathrm{P}^{n-1}$ to be the set of lines through the origin in $\mathbf{R}^n$, with the topology defined in the same way as for $\mathbf{R}\mathrm{P}^2$. These spaces $\mathbf{R}\mathrm{P}^m$ are called **real projective spaces**.

The space $\mathbf{R}\mathrm{P}^0$ is the set of lines through the origin in $\mathbf{R}$. But there is just one such line, so $\mathbf{R}\mathrm{P}^0$ is a single point.

Later on, in Chapter 5, we will see that $\mathbf{R}\mathrm{P}^1$ is topologically the "same" as the circle S^1. This gives you some idea why we write $\mathbf{R}\mathrm{P}^1$ and not $\mathbf{R}\mathrm{P}^2$.

A much more startling fact is that $\mathbf{R}\mathrm{P}^3$ is the "same" as the space SO(3). We will prove this fact in Proposition 5.67.