PART A: NATURAL VIBRATION FREQUENCIES AND MODES

10.1 SYSTEMS WITHOUT DAMPING

Free vibration of linear MDF systems is governed by Eq. (9.2.12) with $\mathbf{p}(t) = \mathbf{0}$, which for systems without damping is

$$\mathbf{m}\ddot{\mathbf{u}} + \mathbf{k}\mathbf{u} = 0 \tag{10.1.1}$$

Equation (10.1.1) represents N homogeneous differential equations that are coupled through the mass matrix, the stiffness matrix, or both matrices; N is the number of DOFs. It is desired to find the solution $\mathbf{u}(t)$ of Eq. (10.1.1) that satisfies the initial conditions

$$\mathbf{u} = \mathbf{u}(0) \qquad \dot{\mathbf{u}} = \dot{\mathbf{u}}(0) \tag{10.1.2}$$

at $t = 0$. A general procedure to obtain the desired solution for any MDF system is developed in Section 10.8. In this section the solution is presented in graphical form that enables us to understand free vibration of an MDF system in qualitative terms.

Figure 10.1.1 shows the free vibration of a two-story shear frame. The story stiffnesses and lumped masses at the floors are noted, and the free vibration is initiated by the deflections shown by curve a in Fig. 10.1.1b. The resulting motion u_j of the two masses is plotted in Fig. 10.1.1d as a function of the time parameter t/T_1, where T_1 is a natural vibration period of the structure, which will be defined later.

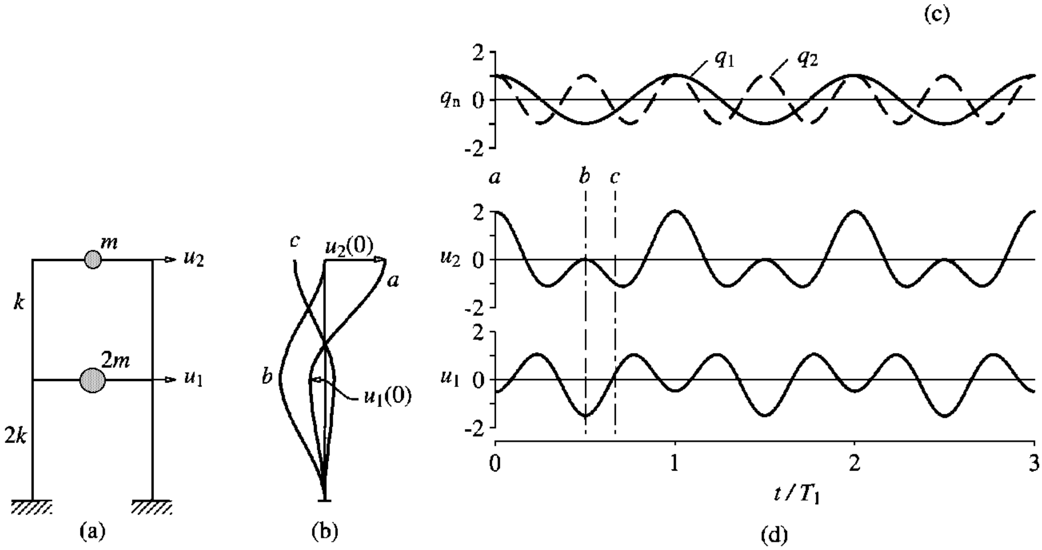

Figure 10.1.1 Free vibration of an undamped system due to arbitrary initial displacement: (a) two-story frame; (b) deflected shapes at time instants a, b, and c; (c) modal coordinates $q_n(t)$; (d) displacement history.

The deflected shapes of the structure at selected time instants a, b, and c are also shown; the $q_n(t)$ plotted in Fig.10.1.1c are discussed in Example 10.11. The displacement–time plot for the jth floor starts with the initial conditions $u_j(0)$ and $\dot{u}_j(0)$; the $u_j(0)$ are identified in Fig. 10.1.1b and $\dot{u}_j(0) = 0$ for both floors. Contrary to what we observed in Fig. 2.1.1 for SDF systems, the motion of each mass (or floor) is not a simple harmonic motion and the frequency of the motion cannot be defined. Furthermore, the deflected shape (i.e., the ratio u_1/u_2) varies with time, as is evident from the differing deflected shapes b and c, which are in turn different from the initial deflected shape a.

An undamped structure would undergo simple harmonic motion without change of deflected shape, however, if free vibration is initiated by appropriate distributions of displacements in the various DOFs. As shown in Figs. 10.1.2 and 10.1.3, two characteristic deflected shapes exist for this two-DOF system such that if it is displaced in one of these shapes and released, it will vibrate in simple harmonic motion, maintaining the initial deflected shape. Both floors reach their extreme displacements at the same time and pass through the equilibrium position at the same time. Observe that the displacements of both floors are in the same direction in the first characteristic deflected shape but in opposite directions in the second characteristic shape. The point of zero displacement, called a *node*,† does not move at all (Fig. 10.1.3); as the mode number n increases, the number of nodes increases accordingly (see Fig. 12.8.2). Each characteristic deflected shape is called a *natural mode of vibration* of an MDF system.

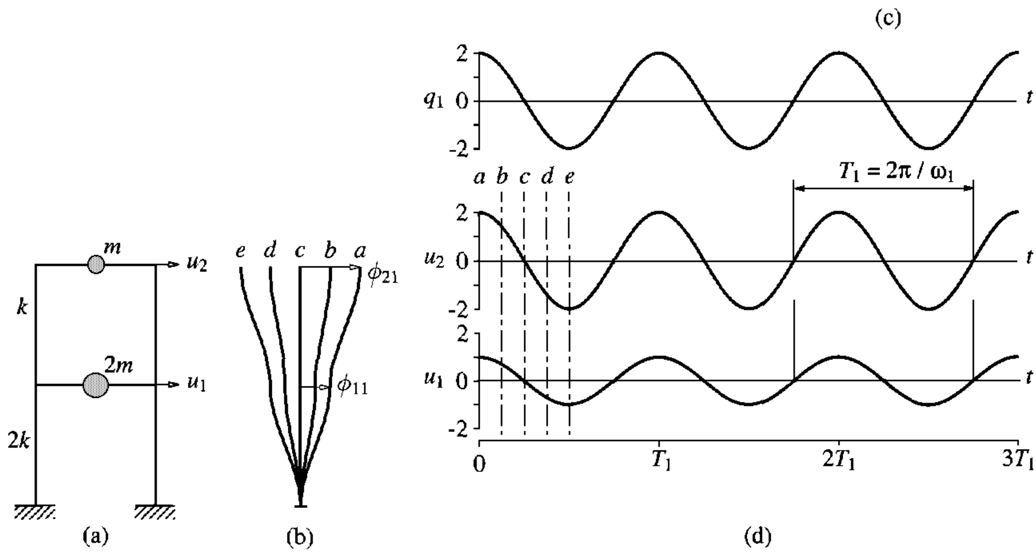

Figure 10.1.2 Free vibration of an undamped system in its first natural mode of vibration: (a) two-story frame; (b) deflected shapes at time instants a, b, c, d, and e; (c) modal coordinate $q_1(t)$; (d) displacement history.

†Recall that we have already used the term *node* for nodal points in the structural idealization; the two different uses of *node* should be clear from the context.

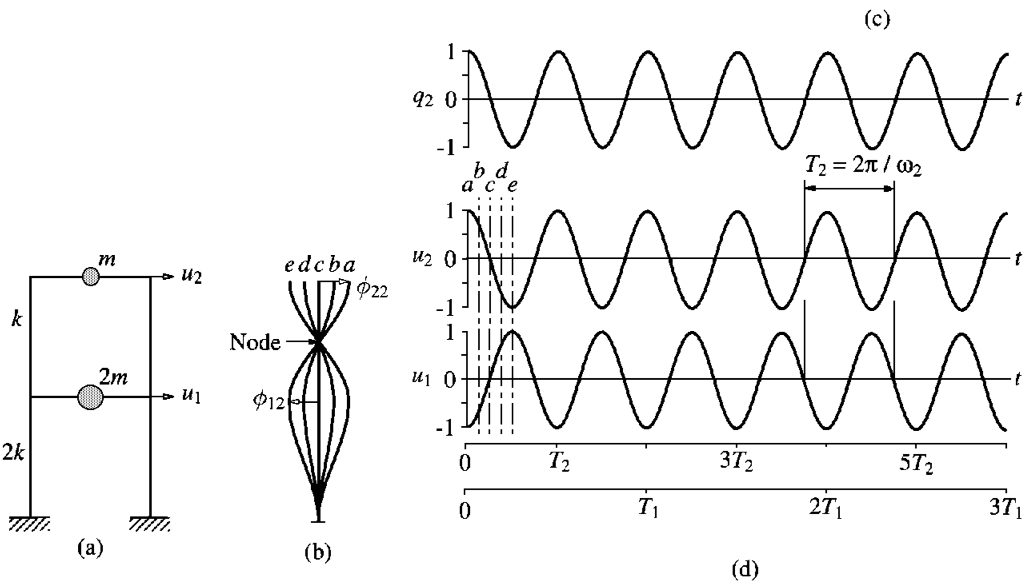

Figure 10.1.3 Free vibration of an undamped system in its second natural mode of vibration: (a) two-story frame; (b) deflected shapes at the time instants a, b, c, d, and e; (c) modal coordinate $q_2(t)$; (d) displacement history.

A *natural period of vibration* T_n of an MDF system is the time required for one cycle of the simple harmonic motion in one of these natural modes. The corresponding *natural circular frequency of vibration* is ω_n and the *natural cyclic frequency of vibration* is f_n, where

$$T_n = \frac{2\pi}{\omega_n} \qquad f_n = \frac{1}{T_n} \tag{10.1.3}$$

Figures 10.1.2 and 10.1.3 show the two natural periods T_n and natural frequencies ω_n ($n = 1, 2$) of the two-story building vibrating in its natural modes $\phi_n = \langle \phi_{1n} \quad \phi_{2n} \rangle^T$. The smaller of the two natural vibration frequencies is denoted by ω_1, and the larger by ω_2. Correspondingly, the longer of the two natural vibration periods is denoted by T_1 and the shorter one as T_2.

10.2 NATURAL VIBRATION FREQUENCIES AND MODES

In this section we introduce the eigenvalue problem whose solution gives the natural frequencies and modes of a system. The free vibration of an undamped system in one of its natural vibration modes, graphically displayed in Figs. 10.1.2 and 10.1.3 for a two-DOF system, can be described mathematically by

$$\mathbf{u}(t) = q_n(t)\phi_n \tag{10.2.1}$$