REVISITING AN OLD PROBLEM WITH STARTLING NEW MATHEMATICAL RESULTS

EQUALITY TESTING AND FERMAT'S THEOREM

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Here I present a simple proof to a famous mathematical problem. The core of the proof is in the use of the *Test for Equality Theorem* -- which is actually partially used in everyday mathematical work in the search of solutions to algebraic equations. My only recommendation is to follow the arguments with patience and with great care.

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It is said that the exception to Fermat's Theorem is this:

$a^2 + b^2 = c^2$; **a**, **b**, **c**, positive numbers [1]

Nevertheless, no mathematical operations can be performed on [1] as it stands. Let's change it a bit. Suppose $a \le b$, and $c > b$. By how much is $c > b$? Let's say by **d**, a positive number, so that

$c = (b + d)$, in which case [1] becomes

$a^2 + b^2 = (b + d)^2 = b^2 + 2bd + d^2$ (highlighted terms cancel each other out), so

$2bd = a^2 – d^2$, and

$b = (a^2 – d^2) / 2d$ [2].

This formula works for every **a**, **d**, positive numbers, $a > d$. In fact, [2] works for all positive rational numbers $a > d$, except $a = b$. For $a \ne b$, [2] represents unambiguously and unequivocally a unique solution on a variable **b** to the equation $a^2 + b^2 = (b + d)^2$.

Notice the change in [2] when we set $a = b$, a perfectly valid condition, since we assumed that $a \le b$, and [2] becomes:

$b = (b^2 – d^2) / 2d$ [2']

The problem with [2'] is that **b** is defined in terms of itself. Nevertheless, [2'] is equivalent to the quadratic equation $b^2 – 2d \cdot b – d^2 = 0$, whose solution $b = d \cdot (1 + \sqrt{2})$ involves *irrational numbers*. [See **Note 1** below].

What happens then, if we make a second substitution in [1], setting $a = (b – f)$, since $a \le b$? Number [1] then becomes

$(b – f)^2 + b^2 = (b + d)^2$ [2].

Is [2] true, i.e. is [2] an *equality*? Let's find out using *equality testing*:

$b^2 – 2bf + f^2 + b^2 = b^2 + 2bd + d^2$ (highlighted terms cancel each other out), or

$b^2 – 2bf + f^2 = 2bd + d^2$, and breaking this relation into two *prospective equalities*, we get:

$b^2 – 2bf = 2bd$, which is true (an *equality*), and

$f^2 = d^2$, which is also true (an *equality*) if $f = d$.

Solving for **b** in $b^2 – 2bf = 2bd$, and collecting terms,

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$b^2 – 2bf – 2bd = 0$, or $b·(b – 2f – 2d) = 0$, and $b = 2f + 2d$, and setting $f = d$, gives

$b = 4d$. Thus, [2] is an *equality*, since it was broken down into two equalities, in accordance with the first part of i) of the *Test for Equality Theorem*, enunciated down below.

Next, let's set $n = 3$ in [2] to determine whether $(b – f)^3 + b^3 = (b + d)^3$ is true (an *equality*):

$b^3 – 3b^2f + 3bf^2 – f^3 + b^3 = b^3 + 3b^2d + 3bd^2 + d^3$ [3], or

$b^3 – 3b^2f + 3bf^2 – f^3 = 3b^2d + 3bd^2 + d^3$ [3']

Again, breaking down [3'] into three *prospective equalities* [a], [b], [c], we test them for equality for $f = d$, as follows:

[a] $b^3 – 3b^2f = 3b^2d$ is true (an *equality*), since

$b^3 – 3b^2f – 3b^2d = 0$, or

$b^2·(b – 3f – 3d) = 0$, and

$b = 3f + 3d$, or $b = 6d$ for $f = d$.

[b] $3bf^2 = 3bd^2$ is true (an *equality*) if $f = d$.

[c] $–f^3 = +d^3$ is false (an *inequality*), even if $f = d$.

Because [a] and [b] are *equalities* and [c] is an *inequality*, [3'] is false (an *inequality*), and also

$(b – f)^3 + b^3 = (b + d)^3$ is false (an *inequality*).

This is *equality testing* at work. We now explain and justify this method.

I'll just say for now that we use *equality testing* all the time, and we don't even notice it. In fact, [3] above, for $f = d$, becomes:

$b^3 – 3b^2d + 3bd^2 – d^3 + b^3 = b^3 + 3b^2d + 3bd^2 + d^3$, which

may be broken down into two *obvious* identities and one *prospective* equality:

- **i)** $3bd^2 = 3bd^2$, an *obvious* identity;
- **ii)** $b^3 = b^3$, an *obvious* identity, and
- **iii)** $b^3 – 3b^2d – d^3 = 3b^2d + d^3$, a *prospective* equality.

To say it differently i), ii), and iii), added together side by side, make up the original relation.

But iii) may itself be broken down into two prospective equalities:

iii[1]] $b^3 – 3b^2d = 3b^2d$, and

iii[2]] $–d^3 = +d^3$.

And [iii[2]] is obviously false (an *inequality*), no matter what. But, $b^3 – 6b^2d = 0$ is true (an *equality*), since

$b^2 \cdot (b - 6d) = 0$, and $b = 6d$.

In short iii] is false (an *inequality*), because iii^1] is true (an *equality*) and iii^2] is false, which means that $(b - f)^3 + b^3 = (b + d)^3$ is false (an *inequality*), as we showed before.

All this by ii) of the Test for Equality Theorem that says:

(x – y + z) ≠ (u + w) if (x – y) = u and z ≠ w or if (x – y) ≠ u and z = w.

Had we used the "normal" procedure with iii] trying to establish its status as an equality or inequality, we would have had to solve it for **b**, as follows:

$b^3 - 3b^2d - d^3 = 3b^2d + d^3$, and $b = \sqrt[3]{6b^2d + 2d^3}$, which is obviously a "definitional loop" that may probably have an *irrational solution.*

What we did before is *testing for equality*, except that we separated from the main relation – we "cancelled out," like we did above – those smaller, evident, and plainly visible identities that came up in the process of expansion, simplification, and reordering of the equation's elements or terms. *Equality testing* proper seeks to establish whether or not a relation is an equality, and doesn't necessarily try to find the equation's solution set.

In essence, *Equality testing*, as presented here and in my book *Non Nobis, Domine*, searches for smaller, concealed, and invisible equalities within larger relations, using a kind of educated or intelligent trial and error methodology.

Now, let's take a similar approach to what we did for **n = 2**, and try to find the general solution to the **prospective equality**:

$a^3 + b^3 = (b + d)^3$, $\quad c = (b + d)$ [4]

$a^3 + b^3 = b^3 + 3b^2d + 3bd^2 + d^3$,

$3b^2d + 3bd^2 = a^3 - d^3$,

$3bd\,(b + d) = a^3 - d^3$,

$b = (a^3 - d^3) / 3d\,(b + d)$ [5]

What worked for **n = 2**, didn't work for **n = 3**, and [5] is not an equality, and can never be an equality that yields a rational solution, since we managed to ***irreducibly*** define **b** in terms of itself. This is another "definitional loop."

Nevertheless, the solution to [5] is the solution to a quadratic equation, mainly

$3d \cdot b^2 + 3d^2 \cdot b - (a^3 - d^3) = 0$, where the coefficients $\mathfrak{a} = 3d$, $\mathfrak{b} = 3d^2$, and $\mathfrak{c} = (a^3 - d^3)$

may be inserted in the quadratic formula.

The previous analytical procedure can be generalized to prove that

$\mathbf{a^n + b^n \neq c^n}$; [7],

a, **b**, **c**, positive numbers; **n > 2** a positive integer.

Setting **a = (b – f)** and **c = (b + d)** above, we prove that

$$\mathbf{(b - f)^n + b^n = (b + d)^n} \quad [7']$$

is false; **0 ≤ f**, **b**, **d**, positive numbers; **n > 2** a positive integer. In other words, we prove that

[7'] is NOT an equality.

Expanding [7'], we get

$$\mathbf{b^n - K_1 b^{(n-1)} f + K_2 b^{(n-2)} f^2 - K_3 b^{(n-3)} f^3 + \ldots \pm f^n + b^n = b^n + K_1 b^{(n-1)} d + K_2 b^{(n-2)} d^2 + K_3 b^{(n-3)} d^3 + \ldots + d^n},$$

$\mathbf{K_i}$ represents the corresponding binomial coefficients, **i = 1**, **2**, **3** …

Setting **f = d**

$$\mathbf{b^n - K_1 b^{(n-1)} d + K_2 b^{(n-2)} d^2 - K_3 b^{(n-3)} d^3 + \ldots \pm d^n = K_1 b^{(n-1)} d + K_2 b^{(n-2)} d^2 + K_3 b^{(n-3)} d^3 \ldots + d^n} \quad [8]$$

Again, highlighted terms cancel each other out, and [8] holds if both

$$\mathbf{b^n - K_1 b^{(n-1)} d = K_1 b^{(n-1)} d} \quad [9]$$

and

$$\mathbf{K_2 b^{(n-2)} d^2 - K_3 b^{(n-3)} d^3 + \ldots \pm d^n = K_2 b^{(n-2)} d^2 + K_3 b^{(n-3)} d^3 \ldots + d^n} \quad [10]$$

are equalities.

Working with [9] we get that

$\mathbf{b^n - 2K_1 b^{(n-1)} d = 0}$, and $\mathbf{b^{(n-1)}(b - 2K_1 d) = 0}$, so $\mathbf{b = 2 \cdot K_1 \cdot d}$. But $\mathbf{K_1 = n}$, so that $\mathbf{b = 2 \cdot n \cdot d}$.

Thus, [9] is an *equality*, since it yields as a solution $\mathbf{b = 2 \cdot n \cdot d}$ for **f = d**, while [10] is not and can never be an equality, because if [10] is indeed an equality, it should be able to be broken down into two or more equalities. In fact, [10] can be broken down into one *equality* – made up by the terms that cancel each other out,

$$\mathbf{K_2 b^{(n-2)} d^2 + K_4 b^{(n-4)} d^4 + \ldots + [d^n] = K_2 b^{(n-2)} d^2 + K_4 b^{(n-4)} d^4 + \ldots + [d^n]} \quad [10']$$

and one inequality – made up of negative terms on the left-hand side of [10] and positive corresponding identical terms on the right-hand side of [10]:

$$\mathbf{- K_3 b^{(n-3)} d^3 - K_5 b^{(n-5)} d^5 \ldots - [d^n] = K_3 b^{(n-3)} d^3 + K_5 b^{(n-5)} d^5 \ldots + [d^n]}, \text{ or}$$

$$\mathbf{- (K_3 b^{(n-3)} d^3 + K_5 b^{(n-5)} d^5 \ldots + [d^n]) = + (K_3 b^{(n-3)} d^3 + K_5 b^{(n-5)} d^5 \ldots + [d^n])} \quad [10'']$$

Note: $\mathbf{d^n}$ will appear in [10'] if **n** is even, while if **n** is odd, $\mathbf{d^n}$ will appear in [10'']. We didn't remove [10'] -- the terms that cancel each other out in [10] -- in order to illustrate the process of *equality testing*.

This proves [10] to be an *inequality* by ii) of the *Test for Equality Theorem*, submitted at the end of this essay. Thus [10] is an *inequality*, while [9] is an *equality*, and by ii) of the *Test for Equality Theorem*, [8] is an *inequality* and its predecessor [7'] is also an *inequality*. Therefore, Fermat's Theorem has been proved.

Like we did before, we can prove [7'] to be false because of "definitional looping":

$\mathbf{a^n + b^n = (b + d)^n}$, and

$\mathbf{a^n + b^n = b^n + K_1 b^{(n-1)} d + K_2 b^{(n-2)} d^2 + K_3 b^{(n-3)} d^3 + \ldots + d^n}$

$\mathbf{K_1 b^{(n-1)} d + K_2 b^{(n-2)} d^2 + K_3 b^{(n-3)} d^3 + \ldots = a^n - d^n}$,

$\mathbf{b \cdot d \cdot (K_1 b^{(n-2)} + K_2 b^{(n-3)} d + K_3 b^{(n-4)} d^2 + \ldots) = a^n - d^n}$

$\mathbf{b = (a^n - d^n) / d \cdot (K_1 b^{(n-2)} + K_2 b^{(n-3)} d + K_3 b^{(n-4)} d^2 + \ldots)}$

Thus, $\mathbf{a^n + b^n \neq c^n}$, since it doesn't have *rational solutions*; **a**, **b**, **c**, positive numbers, $\mathbf{n > 2}$, a positive integer.

I deal with this and other problems in my book called ***Non Nobis Domine***, published by Xlibris Corporation, ISBN 1–4257–0851–X and 1–4257–0850–1.

Now a word about the so-called "definitional loops" that seem to plague this essay. Where do they come from? Let's deal with a simple one:

$\mathbf{b = (b^2 - d^2) / 2d}$ [2'].

By making $\mathbf{a = b}$ we transformed a perfectly good equality, [2], into an inequality, [2']. In other words, the value of **a** may be as "close" as we want to **b**, but never equal to **b** – which means that the *ghost* ***b***, on the right-hand side of the formula may be almost equal to **b**, but always greater than **d**, i.e. $\mathbf{(b^2 - d^2) > 0}$.

Making [2'] into a binomial equation doesn't fix the problem, since the *ghost* ***b*** is only transferred from one notational form of the "equation" to another.

In the case of

$\mathbf{b = (a^3 - d^3) / 3d\,(b + d)}$ [5],

something that comes from an *inequality* and is forcedly assumed to be an *equality*, triggers a mathematical "fix," i.e. the "definitional loop." In other words, the variable **b** on the left-hand side of the equality sign is not the same as the variable **b** inside the formula – let's think of the latter as a *ghost* ***b***, which we may call $\mathbf{b_g}$. By giving a value to $\mathbf{b_g}$, the mathematical engine in the formula produces the value of the variable **b**. In [5] if we make **a = b** the situation gets even more complicated. Again, a binomial notational form of [5] doesn't fix the problem, which means that [2'] and [5] are in fact *inequalities* – the embedded variable is just a red flag that accompanies inequalities of this sort – and their corresponding binomial or polynomial forms are also *inequalities*. In fact, they violate a never known before axiom (or postulate) in mathematics that says something similar to what follows:

A *cannot be equal to* A*, plus or minus something else, all at the same time.* [i]

In symbols, $A \neq A \pm ⅄$, $0 \neq ⅄$ [i]

Of course, we have the tautology **A = A**, which is not explicit enough to exclude the "something else" (or ⅄) above in [i].

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Note 1.

And things get even worse. Take it from the beginning, such that

$b^2 + b^2 = (b + d)^2 = b^2 + 2bd + d^2$, and solving for **b**, **colored in blue**,

$b = \sqrt{(2bd + d^2)}$ [2''],

while solving for **b**, **colored in red**, yields,

$b = (b^2 - d^2) / 2d$, [2']

two different results for seemingly equal variables.

Let's test [2'] and [2''] for equality. Substitute **b** in [2''], **colored in blue**, with the value of **b** in [2'], or $(b^2 - d^2) / 2d = \sqrt{(2bd + d^2)}$. Then solve for

$b = \sqrt{\{2d \cdot \sqrt{(2bd + d^2)} + d^2\}}$,

and the result is a *nested formula*. Solving for **b** inside the square root symbol gives also a *nested formula*, $b = [(b^2 - d^2) / 2d]^2 - d^2] / 2d$. Thus, [2'] is not equivalent to [2''].

But if we substitute the "nested" **b** in [2''] with the value of **b** in [2'], we get

$b = \sqrt{[2d \cdot (b^2 - d^2) / 2d)] + d^2} = b$.

Similarly, substitute the "nested" b in [2'] with the value of b in [2'']:

$b = [(\sqrt{2bd + d^2})^2 - d^2] / 2d = b$,

suggesting that **b** in [2'] and the "nested" **b** in [2'] are different variables, which also seems to be the case for **b** in [2''] and the "nested" **b** in [2''].

In fact, as we explained before, the "nested" **b** in [2'] is just an **a** that approaches **b** from its upper side of **d (** i.e. **a > d)** and never gets to be equal to **b**.

Summarizing, let's change the formulas to

$b = (b_1^2 - d^2) / 2d$ [2'], and

$\mathbf{b} = \sqrt{(2b_2d + d^2)}$ [2'’].

The results are that $\mathbf{b} \neq \mathbf{b}$, $\mathbf{b} = \mathbf{b_1}$, and $\mathbf{b} = \mathbf{b_2}$.

The question then arises, can we solve $\mathbf{b^2 = 2bd + d^2}$ by using intermittingly [2'] and [2'’] above? Which one do we use first, or which is first, the egg or the chicken?

Now a **heuristic** formulation of the *Test for Equality Theorem*:

TEST FOR EQUALITY THEOREM

IF (x – y + z) = (u + w) AND either (x – y) = u or (x – y) = w, THEN, in the first case, z = w or, in the second case, z = u. Also, IF (x – y + z) = (u + w) AND either (z = u) or (z = w), THEN, in the first case, (x – y) = w or, in the second case, (x – y) = u.

(x – y + z) = (u + w), when tested for equality, may be called the *composite relation*, while (x – y) = u and z = w may be called the *component relations*.

The proof of this theorem is trivial. Substitute (x –y) with u and z with w in the *composite relation* above to get (u + w) = (u + w), or substitute (x – y) with w and z with u in the same *composite relation* to get (w + u) = (u + w), which proves the first part of the theorem. Similar substitutions prove the second part.

The following are consequences of this Theorem:

i) (x – y + z) = (u + w) if (x – y) = u and z = w. But (x –y + z) may still be equal to (u + w), even if (x – y) ≠ u and z ≠ w.

ii) (x – y + z) ≠ (u + w) if (x – y) = u and z ≠ w or if (x – y) ≠ u and z = w.

iii) If a relation fails the first part of the test for equality, that is, if (x – y) ≠ u and z ≠ w, it may not fail the second part, and (x – y) = w and z = u may still be true.

iv) If the first part of the test is successful, the second part of the test may also be successful on condition that (x – y) = z = u = w.

v) If (x –y) ≠ z ≠ u ≠ w the test is of no utility, since (x – y + z) may still be equal to (u + w), and the test fails.

Here is another notational form of the Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n,$$

where each $\binom{n}{k}$ is a specific positive integer known as *binomial coefficient.*

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