It turns out that the isomorphism of $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$ is independent of path if and only if the fundamental group is abelian. (See Exercise 3.) This is a stringent requirement on the space X.

Definition. A space X is said to be ***simply connected*** if it is a path-connected space and if $\pi_1(X, x_0)$ is the trivial (one-element) group for some $x_0 \in X$, and hence for every $x_0 \in X$. We often express the fact that $\pi_1(X, x_0)$ is the trivial group by writing $\pi_1(X, x_0) = 0$.

Lemma 52.3. *In a simply connected space X, any two paths having the same initial and final points are path homotopic.*

Proof. Let α and β be two paths from x_0 to x_1. Then $\alpha * \bar{\beta}$ is defined and is a loop on X based at x_0. Since X is simply connected, this loop is path homotopic to the constant loop at x_0. Then

$$[\alpha * \bar{\beta}] * [\beta] = [e_{x_0}] * [\beta],$$

from which it follows that $[\alpha] = [\beta]$. ∎

It is intuitively clear that the fundamental group is a topological invariant of the space X. A convenient way to prove this fact formally is to introduce the notion of the "homomorphism induced by a continuous map."

Suppose that $h : X \to Y$ is a continuous map that carries the point x_0 of X to the point y_0 of Y. We often denote this fact by writing

$$h : (X, x_0) \longrightarrow (Y, y_0).$$

If f is a loop in X based at x_0, then the composite $h \circ f : I \to Y$ is a loop in Y based at y_0. The correspondence $f \to h \circ f$ thus gives rise to a map carrying $\pi_1(X, x_0)$ into $\pi_1(Y, y_0)$. We define it formally as follows:

h_*

Definition. Let $h : (X, x_0) \to (Y, y_0)$ be a continuous map. Define

$$h_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$$

by the equation

$$h_*([f]) = [h \circ f].$$

The map h_* is called the ***homomorphism induced by h***, relative to the base point x_0.

The map h_* is well-defined, for if F is a path homotopy between the paths f and f', then $h \circ F$ is a path homotopy between the paths $h \circ f$ and $h \circ f'$. The fact that h_* is a homomorphism follows from the equation

$$(h \circ f) * (h \circ g) = h \circ (f * g).$$

4. Consider the covering map $p : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2 - \mathbf{0}$ of Example 6 of §53. Find liftings of the paths

$$f(t) = (2 - t, 0),$$
$$g(t) = ((1 + t)\cos 2\pi t, (1 + t)\sin 2\pi t)$$
$$h(t) = f * g.$$

Sketch these paths and their liftings.

5. Consider the covering map $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ of Example 4 of §53. Consider the path

$$f(t) = (\cos 2\pi t, \sin 2\pi t) \times (\cos 4\pi t, \sin 4\pi t)$$

in $S^1 \times S^1$. Sketch what f looks like when $S^1 \times S^1$ is identified with the doughnut surface D. Find a lifting $\tilde{f}$ of f to $\mathbb{R} \times \mathbb{R}$, and sketch it.

6. Consider the maps $g, h : S^1 \to S^1$ given $g(z) = z^n$ and $h(z) = 1/z^n$. (Here we represent S^1 as the set of complex numbers z of absolute value 1.) Compute the induced homomorphisms g_*, h_* of the infinite cyclic group $\pi_1(S^1, b_0)$ into itself. [*Hint:* Recall the equation $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.]

7. Generalize the proof of Theorem 54.5 to show that the fundamental group of the torus is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.

8. Let $p : E \to B$ be a covering map, with E path connected. Show that if B is simply connected, then p is a homeomorphism.

§55 Retractions and Fixed Points

We now prove several classical results of topology that follow from our knowledge of the fundamental group of S^1.

Definition. If $A \subset X$, a ***retraction*** of X onto A is a continuous map $r : X \to A$ such that $r|A$ is the identity map of A. If such a map r exists, we say that A is a ***retract*** of X.

Lemma 55.1. *If A is a retract of X, then the homomorphism of fundamental groups induced by inclusion $j : A \to X$ is injective.*

Proof. If $r : X \to A$ is a retraction, then the composite map $r \circ j$ equals the identity map of A. It follows that $r_* \circ j_*$ is the identity map of $\pi_1(A, a)$, so that j_* must be injective. ■

Theorem 55.2 (No-retraction theorem). *There is no retraction of B^2 onto S^1.*

Proof. If S^1 were a retract of B^2, then the homomorphism induced by inclusion $j : S^1 \to B^2$ would be injective. But the fundamental group of S^1 is nontrivial and the fundamental group of B^2 is trivial. ■