Force on a body depending on the trajectory without explicit time dependence

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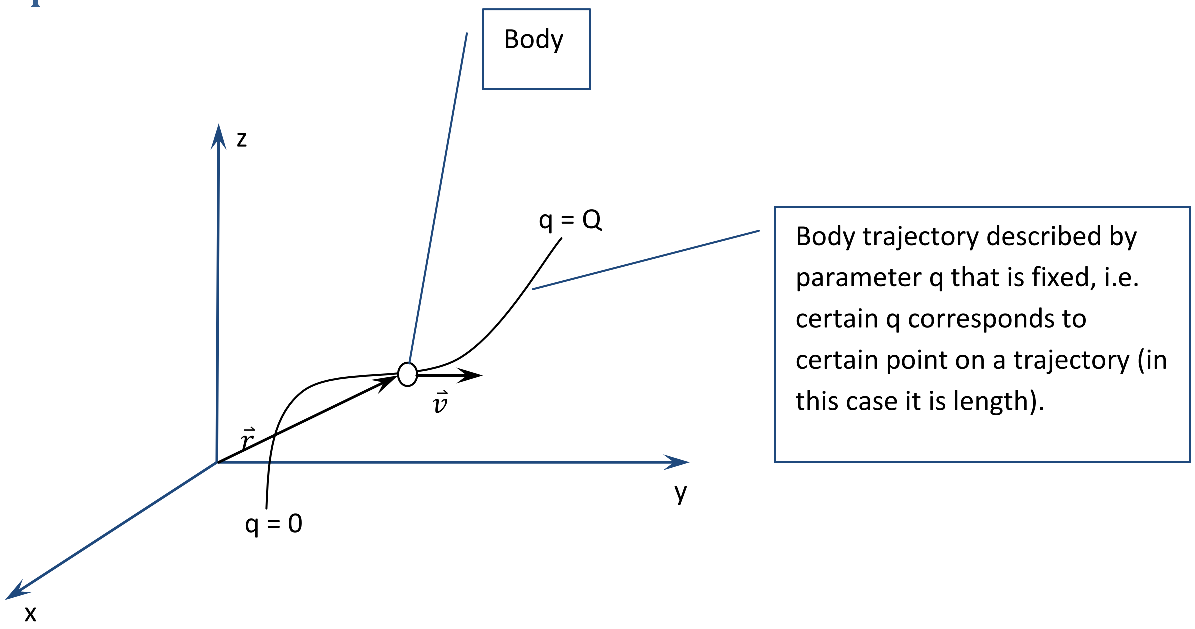
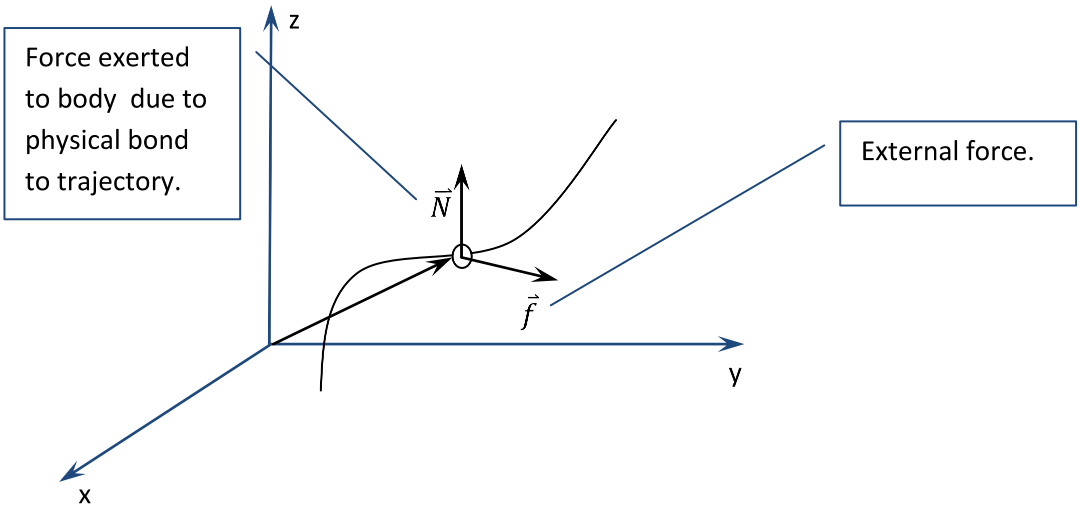
Setup



$$|d\vec{r}| = dq$$

If there are external forces (but no friction) than the system can be described by:



$$m\,\ddot{\vec{r}} = \vec{f} + \vec{N}$$

or

$$\vec{N} = m\,\ddot{\vec{r}} - \vec{f}$$

where

$$\vec{f} = -\vec{\nabla}U$$

Lagrangian

For the system there is also Lagrangian

$$L = \frac{1}{2}\, m\, \dot{q}^2 - U(q)$$

and from

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

we obtain

$$-\frac{dU}{dq} - m\, \ddot{q} = 0$$

or

$$m\, \ddot{q} = -\frac{dU}{dq} = -\frac{d\vec{r}}{dq} \cdot \vec{\nabla} U = \frac{d\vec{r}}{dq} \cdot \vec{f}$$

from where we can solve for

$$q(t)$$

$$\dot{q}(q)$$

$$\ddot{q}(q)$$

where speed and acceleration are dependent only of position.

Bonding force

Next we need to determine bonding force

$$\vec{N} = m\,\ddot{\vec{r}} - \vec{f}$$

To do so we can use

$$\dot{\vec{r}} = \dot{q}\,\frac{d\vec{r}}{dq}$$

$$\ddot{\vec{r}} = \ddot{q}\,\frac{d\vec{r}}{dq} + \dot{q}^2\,\frac{d^2\vec{r}}{dq^2}$$

and so

$$\vec{N} = -\vec{f} + m\,\ddot{q}\,\frac{d\vec{r}}{dq} + m\,\dot{q}^2\,\frac{d^2\vec{r}}{dq^2}$$

Final set of equations

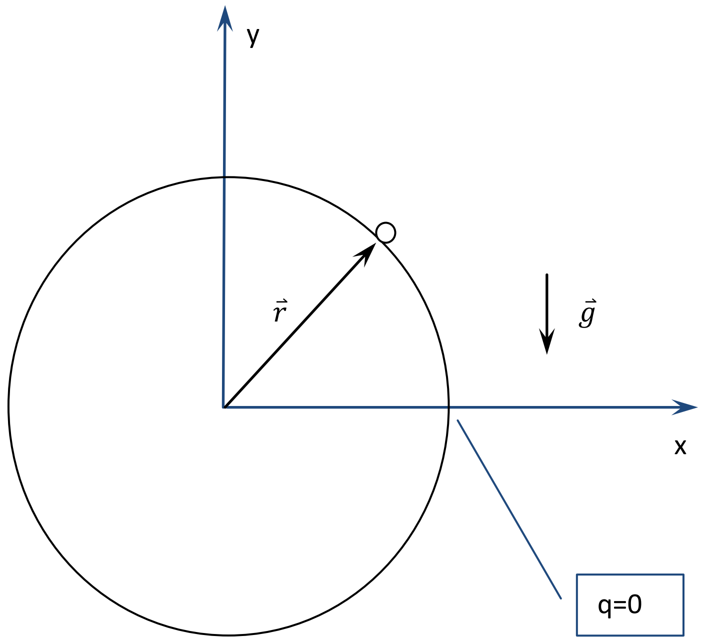
So in short we have two equations

$$m\,\ddot{q} = \frac{d\vec{r}}{dq}\cdot\vec{f}$$

$$\vec{N} = -\vec{f} + m\,\ddot{q}\,\frac{d\vec{r}}{dq} + m\,\dot{q}^2\,\frac{d^2\vec{r}}{dq^2}$$

First determines the connection of speed and acceleration dependent only of position, and not on time explicitly, and the second determines the force depending on the position on trajectory.

Example: Motion of a body in gravitation and bounded on a circle



Starting from

$$m\,\ddot{q} = \frac{d\vec{r}}{dq}\cdot\vec{f}$$

$$\vec{N} = -\vec{f} + m\,\ddot{q}\,\frac{d\vec{r}}{dq} + m\,\dot{q}^2\,\frac{d^2\vec{r}}{dq^2}$$

where parametric description of trajectory is

$$\vec{r} = \hat{x}R\cos\left(\frac{q}{R}\right) + \hat{y}R\sin\left(\frac{q}{R}\right)$$

and

$$\vec{f} = -\hat{y}mg$$

we have

$$m\,\ddot{q} = \frac{d\vec{r}}{dq}\cdot\vec{f} = -\,mg\cos\left(\frac{q}{R}\right)$$

$$\ddot{q} = -\,g\cos\left(\frac{q}{R}\right)$$

$$\dot{q}\,\frac{d\dot{q}}{dq} = -\,g\cos\left(\frac{q}{R}\right)$$

$$\dot{q}^2 = -\,2Rg\sin\left(\frac{q}{R}\right) + \dot{q}^2(0) + 2Rg\sin\left(\frac{q(0)}{R}\right)$$

On the other hand we have

$$\frac{d\vec{r}}{dq} = -\hat{x}\sin\left(\frac{q}{R}\right) + \hat{y}\cos\left(\frac{q}{R}\right)$$

$$\frac{d^2\vec{r}}{dq^2} = -\hat{x}\frac{1}{R}\cos\left(\frac{q}{R}\right) - \hat{y}\frac{1}{R}\sin\left(\frac{q}{R}\right)$$

so the bonding force is now

$$\vec{N} = -\vec{f} + m\,\ddot{q}\,\frac{d\vec{r}}{dq} + m\,\dot{q}^2\,\frac{d^2\vec{r}}{dq^2}$$

$$\vec{N} = \hat{y}mg - \;mg\cos\left(\frac{q}{R}\right)\left(-\hat{x}\sin\left(\frac{q}{R}\right) + \hat{y}\cos\left(\frac{q}{R}\right)\right) +$$

$$+m\left(-\,2Rg\sin\left(\frac{q}{R}\right) + \dot{q}^2(0) + 2Rg\sin\left(\frac{q(0)}{R}\right)\right)\left(-\hat{x}\frac{1}{R}\cos\left(\frac{q}{R}\right) - \hat{y}\frac{1}{R}\sin\left(\frac{q}{R}\right)\right)$$

$$\vec{N} = \;\hat{x}\left(mg\sin\left(\frac{q}{R}\right)\cos\left(\frac{q}{R}\right) - m\frac{1}{R}\cos\left(\frac{q}{R}\right)\left(-\,2Rg\sin\left(\frac{q}{R}\right) + \dot{q}^2(0) + 2Rg\sin\left(\frac{q(0)}{R}\right)\right)\right)$$

$$+\hat{y}\left(mg - mg\cos\left(\frac{q}{R}\right)\cos\left(\frac{q}{R}\right) - m\,\frac{1}{R}\sin\left(\frac{q}{R}\right)\left(-\,2Rg\sin\left(\frac{q}{R}\right) + \dot{q}^2(0) + 2Rg\sin\left(\frac{q(0)}{R}\right)\right)\right)$$

$$\vec{N} = \;\hat{x}\left(3mg\sin\left(\frac{q}{R}\right)\cos\left(\frac{q}{R}\right) - m\frac{1}{R}\cos\left(\frac{q}{R}\right)\left(\dot{q}^2(0) + 2Rg\sin\left(\frac{q(0)}{R}\right)\right)\right)$$

$$+\hat{y}\left(3mg\sin^2\left(\frac{q}{R}\right) - m\,\frac{1}{R}\sin\left(\frac{q}{R}\right)\left(\dot{q}^2(0) + 2Rg\sin\left(\frac{q(0)}{R}\right)\right)\right)$$

If we now put the initial conditions (for example)

$$q(0) = 0$$

$$\dot{q}(0) = v$$

the force becomes dependent on trajectory only

$$\vec{N} = \;\hat{x}\left(3mg\sin\left(\frac{q}{R}\right)\cos\left(\frac{q}{R}\right) - m\frac{v^2}{R}\cos\left(\frac{q}{R}\right)\right)$$

$$+\hat{y}\left(3mg\sin^2\left(\frac{q}{R}\right) - m\,\frac{v^2}{R}\sin\left(\frac{q}{R}\right)\right)$$

Special case 1: force at maximal position

$$q = \frac{\pi}{2} R$$

$$\vec{N} = \hat{y}\left(3mg - m\,\frac{v^2}{R}\right)$$

That could seems strange, but we need to recalculate the velocity to instant velocity at the maximal position u (can do it over energy)

$$\frac{1}{2}\,m\,v^2 = \frac{1}{2}\,m\,u^2 + mgR$$

$$\frac{m\,v^2}{R} = \frac{m\,u^2}{R} + 2mg$$

From where

$$\vec{N} = \hat{y}\left(mg - \frac{m\,u^2}{R}\right)$$

That resembles something familiar.

Special case 2: force at minimal position

$$q = -\frac{\pi}{2}R$$

$$\vec{N} = \hat{y}\left(3mg + m\,\frac{v^2}{R}\right)$$

again we can recalculate the velocity to instant velocity at the minimal position w (can do it over energy)

$$\frac{1}{2}\,m\,v^2 = \frac{1}{2}\,m\,w^2 - mgR$$

$$\frac{m\,v^2}{R} = \frac{m\,w^2}{R} - 2mg$$

From where

$$\vec{N} = \hat{y}\left(mg + \frac{m\,w^2}{R}\right)$$