Example 7.6. The gambler has initial capital a and plays at unit stakes until his capital increases to c $(0 \le a \le c)$ or he is ruined. Here $F_0 = a$ and $W_n = 1$, and so $F_n = a + S_n$. The policy is bounded by c and F_τ is c or 0 according as the gambler succeeds or fails. If $p = \frac{1}{2}$ and if s is the probability of success, then $a = F_0 = E[F_\tau] = sc$. Thus $s = a/c$. This gives a new derivation of (7.7) for the case $p = \frac{1}{2}$. The argument assumes however that play is certain to terminate. If $p \le \frac{1}{2}$, Theorem 7.2 only gives $s \le a/c$, which is weaker than (7.7). ■

Example 7.7. Suppose as before that $F_0 = a$ and $W_n = 1$, so that $F_n = a + S_n$, but suppose the stopping rule is to quit as soon as F_n reaches $a + b$. Here F_n^* is bounded above by $a + b$ but is not bounded below. If $p = \frac{1}{2}$, the gambler is by (7.8) certain to achieve his goal, so that $F_\tau = a + b$. In this case $F_0 = a < a + b = E[F_\tau]$. This illustrates the effect of infinite capital. It also illustrates the need for uniform boundedness in Theorem 5.3 (compare Example 5.6). ■

For some other systems (gamblers call them "martingales"), see the problems. For most such systems there is a large chance of a small gain and a small chance of a large loss.

Bold Play*

The formula (7.7) gives the chance that a gambler betting unit stakes can increase his fortune from a to c before being ruined. Suppose that a and c happen to be even and that at each trial the wager is two units instead of one. Since this has the effect of halving a and c, the chance of success is now

$$\frac{\rho^{a/2} - 1}{\rho^{c/2} - 1} = \frac{\rho^a - 1}{\rho^c - 1}\,\frac{\rho^{c/2} + 1}{\rho^{a/2} + 1}, \qquad \frac{q}{p} = \rho \ne 1.$$

If $\rho > 1$ $(p < \frac{1}{2})$, the second factor on the right exceeds 1: Doubling the stakes increases the probability of success in the unfavorable case $\rho > 1$. In case $\rho = 1$, the probability remains the same.

There is a sense in which large stakes are optimal. It will be convenient to rescale so that the initial fortune satisfies $0 \le F_0 \le 1$ and the goal is 1. The policy of *bold play* is this: At each stage the gambler bets his entire fortune, unless a win would carry him past his goal of 1, in which case he bets just

* This topic may be omitted.

enough that a win would exactly achieve that goal:

$$W_n = \begin{cases} F_{n-1} & \text{if } 0 \le F_{n-1} \le \frac{1}{2}, \\ 1 - F_{n-1} & \text{if } \frac{1}{2} \le F_{n-1} \le 1. \end{cases} \tag{7.25}$$

(It is convenient to allow even irrational fortunes.) As for stopping, the policy is to quit as soon as F_n reaches 0 or 1.

Suppose that play has not terminated by time $k-1$; under the policy (7.25), if play is not to terminate at time k, then X_k must be $+1$ or -1 according as $F_{k-1} \le \frac{1}{2}$ or $F_{k-1} \ge \frac{1}{2}$, and the conditional probability of this is at most $m = \max\{p, q\}$. It follows by induction that the probability that bold play continues beyond time n is at most m^n, and so play is certain to terminate (τ is finite with probability 1).

It will be shown that in the subfair case, bold play maximizes the probability of successfully reaching the goal of 1. This is the *Dubins–Savage theorem*. It will further be shown that there are other policies that are also optimal in this sense, and this maximum probability will be calculated. Bold play can be substantially better than betting at constant stakes. This contrasts with Theorems 7.1 and 7.2 concerning respects in which gambling systems are worthless.

From now on, consider only policies π that are bounded by 1 (see (7.24)). Suppose further that play stops as soon as F_n reaches 0 or 1 and that this is certain eventually to happen. Since F_τ assumes the values 0 and 1, and since $[F_\tau = x] = \bigcup_{n=0}^{\infty} [\tau = n] \cap [F_n = x]$ for $x = 0$ and $x = 1$, F_τ is a simple random variable. Bold play is one such policy π.

The policy π leads to success if $F_\tau = 1$. Let $Q_\pi(x)$ be the probability of this for an initial fortune $F_0 = x$:

$$Q_\pi(x) = P[F_\tau = 1], \qquad F_0 = x. \tag{7.26}$$

Since F_n is a function $\psi_n(F_0, X_1(\omega), \ldots, X_n(\omega)) = \Psi_n(F_0, \omega)$, (7.26) in expanded notation is $Q_\pi(x) = P[\omega\colon \Psi_{\tau(x,\omega)}(x, \omega) = 1]$. As π specifies that play stops at the boundaries 0 and 1,

$$\begin{cases} Q_\pi(0) = 0, & Q_\pi(1) = 1, \\ 0 \le Q_\pi(x) \le 1, & 0 \le x \le 1. \end{cases} \tag{7.27}$$

Let Q be the Q_π for bold play. (The notation does not show the dependence of Q and Q_π on p, which is fixed.)

Theorem 7.3. *In the subfair case, $Q_\pi(x) \le Q(x)$ for all π and all x.*

PROOF. Under the assumption $p \le q$, it will be shown later that

$$Q(x) \ge pQ(x+t) + qQ(x-t), \qquad 0 \le x - t \le x \le x + t \le 1. \tag{7.28}$$

This can be interpreted as saying that the chance of success under bold play starting at x is at least as great as the chance of success if the amount t is wagered and bold play then pursued from $x + t$ in case of a win and from $x - t$ in case of a loss. Under the assumption of (7.28), optimality can be proved as follows.

Consider a policy π, and let F_n and F_n^* be the simple random variables defined by (7.14) and (7.19) for *this policy*. Now $Q(x)$ is a real function, and so $Q(F_n^*)$ is also a simple random variable; it can be interpreted as the conditional chance of success if π is replaced by bold play after time n. By (7.20), $F_n^* = x + tX_n$ if $F_{n-1}^* = x$ and $W_n^* = t$. Therefore,

$$Q(F_n^*) = \sum_{x,t} I_{[F_{n-1}^* = x, W_n^* = t]} Q(x + tX_n),$$

where x and t vary over the (finite) ranges of F_{n-1}^* and W_n^*, respectively.

For each x and t, the indicator above is measurable $\mathscr{F}_{n-1}$ and $Q(x + tX_n)$ is measurable $\sigma(X_n)$; since the X_n are independent, (5.21) and (5.14) give

$$E[Q(F_n^*)] = \sum_{x,t} P[F_{n-1}^* = x, W_n^* = t] E[Q(x + tX_n)]. \tag{7.29}$$

By (7.28), $E[Q(x + tX_n)] \le Q(x)$ if $0 \le x - t \le x \le x + t \le 1$. As it is assumed of π that F_n^* lies in [0, 1] (that is, $W_n^* \le \min\{F_{n-1}^*, 1 - F_{n-1}^*\}$), the probability in (7.29) is 0 unless x and t satisfy this constraint. Therefore,

$$\begin{aligned} E[Q(F_n^*)] &\le \sum_{x,t} P[F_{n-1}^* = x, W_n^* = t] Q(x) \\ &= \sum_{x} P[F_{n-1}^* = x] Q(x) = E[Q(F_{n-1}^*)]. \end{aligned}$$

This is true for each n, and so $E[Q(F_n^*)] \le E[Q(F_0^*)] = Q(F_0)$. Since $Q(F_n^*) = Q(F_\tau)$ for $n \ge \tau$, Theorem 5.3 implies that $E[Q(F_\tau)] \le Q(F_0)$. Since $x = 1$ implies that $Q(x) = 1$, $P[F_\tau = 1] \le E[Q(F_\tau)] \le Q(F_0)$. Thus $Q_\pi(F_0) \le Q(F_0)$ for the policy π, whatever F_0 may be.

It remains to analyze Q and prove (7.28). Everything hinges on the functional equation

$$Q(x) = \begin{cases} pQ(2x), & 0 \le x \le \frac{1}{2}, \\ p + qQ(2x-1), & \frac{1}{2} \le x \le 1. \end{cases} \tag{7.30}$$

For $x = 0$ and $x = 1$ this is obvious because $Q(0) = 0$ and $Q(1) = 1$. The idea is this: Suppose that the initial fortune is x. If $x \le \frac{1}{2}$, the first stake under bold play is x; if the gambler is to succeed in reaching 1, he must win the first trial (probability p) and then from his new fortune $x + x = 2x$ go on to succeed (probability $Q(2x)$); this makes the first half of (7.30) plausible. If $x \ge \frac{1}{2}$, the first stake is $1 - x$; the gambler can succeed either by winning the first trial (probability p) or by losing the first trial (probability q) and then going on from his new fortune $x - (1 - x) = 2x - 1$ to succeed (probability $Q(2x - 1)$); this makes the second half of (7.30) plausible.

It is also intuitively clear that $Q(x)$ must be an increasing function of x $(0 \le x \le 1)$: the more money the gambler starts with, the better off he is. Finally, it is intuitively clear that $Q(x)$ ought to be a continuous function of the initial fortune x.

A formal proof of (7.30) can be constructed as for the difference equation (7.5). If $\beta(x)$ is x for $x \le \frac{1}{2}$ and $1 - x$ for $x \ge \frac{1}{2}$, then under bold play $W_n = \beta(F_{n-1})$. Starting from $f_0(x) = x$, recursively define

$$f_n(x; x_1, \dots, x_n) = f_{n-1}(x; x_1, \dots, x_{n-1}) + \beta(f_{n-1}(x; x_1, \dots, x_{n-1}))x_n.$$

Then $F_n = f_n(F_0; X_1, \dots, X_n)$. Now define

$$g_n(x; x_1, \dots, x_n) = \max_{0 \le k \le n} f_k(x; x_1, \dots, x_k).$$

If $F_0 = x$, then $T_n(x) = [g_n(x; X_1, \dots, X_n) = 1]$ is the event that bold play will by time n successfully increase the gambler's fortune to 1. From the recursive definition it follows by induction on n that for $n \ge 1$, $f_n(x; x_1, \dots, x_n) = f_{n-1}(x + \beta(x)x_1; x_2, \dots, x_n)$ and hence that $g_n(x; x_1, \dots, x_n) = \max\{x, g_{n-1}(x + \beta(x)x_1; x_2, \dots, x_n)\}$. Since $x = 1$ implies $g_{n-1}(x + \beta(x)x_1; x_2, \dots, x_n) \ge x + \beta(x)x_1 = 1$, $T_n(x) = [g_{n-1}(x + \beta(x)X_1; X_2, \dots, X_n) = 1]$, and since the X_i are independent and identically distributed, $P(T_n(x)) = P([X_1 = +1] \cap T_n(x)) + P([X_1 = -1] \cap T_n(x)) = pP[g_{n-1}(x + \beta(x); X_2, \dots, X_n) = 1] + qP[g_{n-1}(x - \beta(x); X_2, \dots, X_n) = 1] = pP(T_{n-1}(x + \beta(x))) + qP(T_{n-1}(x - \beta(x)))$. Letting $n \to \infty$ now gives $Q(x) = pQ(x + \beta(x)) + qQ(x - \beta(x))$, which reduces to (7.30) because $Q(0) = 0$ and $Q(1) = 1$.

Suppose that $y = f_{n-1}(x; x_1, \dots, x_{n-1})$ is nondecreasing in x. If $x_n = +1$, then $f_n(x; x_1, \dots, x_n)$ is $2y$ if $0 \le y \le \frac{1}{2}$ and 1 if $\frac{1}{2} \le y \le 1$; if $x_n = -1$, then

$f_n(x; x_1, \dots, x_n)$ is 0 if $0 \le y \le \frac{1}{2}$ and $2y-1$ if $\frac{1}{2} \le y \le 1$. In any case, $f_n(x; x_1, \dots, x_n)$ is also nondecreasing in x, and by induction this is true for every n. It follows that the same is true of $g_n(x; x_1, \dots, x_n)$, of $P(T_n(x))$, and of $Q(x)$. Thus $Q(x)$ is nondecreasing.

Since $Q(1) = 1$, (7.30) implies that $Q(\frac{1}{2}) = pQ(1) = p$, $Q(\frac{1}{4}) = pQ(\frac{1}{2}) = p^2$, $Q(\frac{3}{4}) = p + qQ(\frac{1}{2}) = p + pq$. More generally, if $p_0 = p$ and $p_1 = q$, then

$$(7.31)\quad Q\left(\frac{k}{2^n}\right) = \sum\left[p_{u_1} \cdots p_{u_n} \colon \sum_{i=1}^{n} \frac{u_i}{2^i} < \frac{k}{2^n}\right], \qquad 0 < k \le 2^n, \qquad n \ge 1,$$

the sum extending over n-tuples $(u_1, \dots, u_n)$ of 0's and 1's satisfying the condition indicated. Indeed, it is easy to see that (7.31) is the same thing as

$$(7.32)\qquad Q(.u_1 \cdots u_n + 2^{-n}) - Q(.u_1 \cdots u_n) = p_{u_1} p_{u_2} \cdots p_{u_n}$$

for each dyadic rational $.u_1 \cdots u_n$ of rank n. If $.u_1 \cdots u_n + 2^{-n} \le \frac{1}{2}$, then $u_1 = 0$ and by (7.30) the difference in (7.32) is $p_0[Q(.u_2 \cdots u_n + 2^{-n+1}) - Q(.u_2 \cdots u_n)]$. But (7.32) follows inductively from this and a similar relation for the case $.u_1 \cdots u_n \ge \frac{1}{2}$.

Therefore $Q(k2^{-n}) - Q((k-1)2^{-n})$ is bounded by $\max\{p^n, q^n\}$, and so by monotonicity Q is continuous. Since (7.32) is positive, it follows that Q is strictly increasing over $[0,1]$.

Thus Q is continuous and increasing and satisfies (7.30). The inequality (7.28) is still to be proved. It is equivalent to the assertion that

$$\Delta(r,s) = Q(a) - pQ(s) - qQ(r) \ge 0$$

if $0 \le r \le s \le 1$, where a stands for the average: $a = \frac{1}{2}(r+s)$. Since Q is continuous, it suffices to prove the inequality for r and s of the form $k/2^n$, and this will be done by induction on n. Checking all cases disposes of $n = 0$. Assume that the inequality holds for a particular n, and that r and s have the form $k/2^{n+1}$. There are four cases to consider.

CASE 1. $s \le \frac{1}{2}$. By the first part of (7.30), $\Delta(r,s) = p\Delta(2r,2s)$. Since $2r$ and $2s$ have the form $k/2^n$, the induction hypothesis implies that $\Delta(2r,2s) \ge 0$.

CASE 2. $\frac{1}{2} \le r$. By the second part of (7.30),

$$\Delta(r,s) = q\Delta(2r-1, 2s-1) \ge 0.$$

CASE 3. $r \le a \le \frac{1}{2} \le s$. By (7.30),

$$\Delta(r,s) = pQ(2a) - p[p + qQ(2s-1)] - q[pQ(2r)].$$

Since $\frac{1}{2} \le s \le r + s = 2a \le 1$, $Q(2a) = p + qQ(4a - 1)$; since $0 \le 2a - \frac{1}{2} \le \frac{1}{2}$, $Q(2a - \frac{1}{2}) = pQ(4a - 1)$. Therefore, $pQ(2a) = p^2 + qQ(2a - \frac{1}{2})$, and it follows that

$$\Delta(r,s) = q\left[Q(2a - \tfrac{1}{2}) - pQ(2s - 1) - pQ(2r)\right].$$

Since $p \le q$, the right side does not increase if either of the two p's is changed to q. Hence

$$\Delta(r,s) \ge q \max\left[\Delta(2r, 2s - 1), \Delta(2s - 1, 2r)\right].$$

The induction hypothesis applies to $2r \le 2s - 1$ or to $2s - 1 \le 2r$, as the case may be, and so one of the two Δ's on the right is nonnegative.

CASE 4. $r \le \frac{1}{2} \le a \le s$. By (7.30),

$$\Delta(r,s) = pq + qQ(2a - 1) - pqQ(2s - 1) - pqQ(2r).$$

Since $0 \le 2a - 1 = r + s - 1 \le \frac{1}{2}$, $Q(2a - 1) = pQ(4a - 2)$; since $\frac{1}{2} \le 2a - \frac{1}{2} = r + s - \frac{1}{2} \le 1$, $Q(2a - \frac{1}{2}) = p + qQ(4a - 2)$. Therefore, $qQ(2a - 1) = pQ(2a - \frac{1}{2}) - p^2$, and it follows that

$$\Delta(r,s) = p\left[q - p + Q(2a - \tfrac{1}{2}) - qQ(2s - 1) - qQ(2r)\right].$$

If $2s - 1 \le 2r$, the right side here is

$$p\left[(q - p)(1 - Q(2r)) + \Delta(2s - 1, 2r)\right] \ge 0.$$

If $2r \le 2s - 1$, the right side is

$$p\left[(q - p)(1 - Q(2s - 1)) + \Delta(2r, 2s - 1)\right] \ge 0.$$

This completes the proof of (7.28) and hence of Theorem 7.3. ■

The equation (7.31) has an interesting interpretation. Let $Z_1, Z_2, \ldots$ be independent random variables satisfying $P[Z_n = 0] = p_0 = p$ and $P[Z_n = 1] = p_1 = q$. From $P[Z_n = 1 \text{ i.o.}] = 1$ and $\sum_{i > n} Z_i 2^{-i} \le 2^{-n}$ it follows that $P[\sum_{i=1}^{\infty} Z_i 2^{-i} \le k2^{-n}] \le P[\sum_{i=1}^{n} Z_i 2^{-i} < k2^{-n}] \le P[\sum_{i=1}^{\infty} Z_i 2^{-i} \le k2^{-n}]$. Since by (7.31) the middle term is $Q(k2^{-n})$.

$$Q(x) = P\left[\sum_{i=1}^{\infty} Z_i 2^{-i} \le x\right] \tag{7.33}$$

holds for dyadic rational x and hence by continuity holds for all x. In Section 31, Q will reappear as a continuous, strictly increasing function singular in the sense of Lebesgue. On p. 428 is a graph for the case $p_0 = .25$.

Note that $Q(x) \equiv x$ in the fair case $p = \frac{1}{2}$. In fact, for a bounded policy Theorem 7.2 implies that $E[F_\tau] = F_0$ in the fair case, and if the policy is to stop as soon as the fortune reaches 0 or 1, then the chance of successfully reaching 1 is $P[F_\tau = 1] = E[F_\tau] = F_0$. Thus in the fair case with initial fortune x, the chance of success is x for *every* policy that stops at the boundaries, and x is an upper bound even if stopping earlier is allowed.

Example 7.8. The gambler of Example 7.1 has capital \$900 and goal \$1000. For a fair game ($p = \frac{1}{2}$) his chance of success is .9 whether he bets unit stakes or adopts bold play. At red-and-black ($p = \frac{18}{38}$), his chance of success with unit stakes is .00003; an approximate calculation based on (7.31) shows that under bold play his chance $Q(.9)$ of success increases to about .88, which compares well with the fair case. ∎

Example 7.9. In Example 7.2 the capital is \$100 and the goal \$20,000. At unit stakes the chance of successes is .005 for $p = \frac{1}{2}$ and 3×10^{-911} for $p = \frac{18}{38}$. Another approximate calculation shows that bold play at red-and-black gives the gambler probability about .003 of success, which again compares well with the fair case.

This example illustrates the point of Theorem 7.3. The gambler enters the casino knowing that he must by dawn convert his \$100 into \$20,000 or face certain death at the hands of criminals to whom he owes that amount. Only red-and-black is available to him. The question is not whether to gamble—he *must* gamble. The question is how to gamble so as to maximize the chance of survival, and bold play is the answer. ∎

There are policies other than the bold one which achieve the maximum success probability $Q(x)$. Suppose that as long as the gambler's fortune x is less than $\frac{1}{2}$ he bets x for $x \le \frac{1}{4}$ and $\frac{1}{2} - x$ for $\frac{1}{4} \le x \le \frac{1}{2}$. This is, in effect, the bold-play strategy scaled down to the interval $[0, \frac{1}{2}]$, and so the chance he ever reaches $\frac{1}{2}$ is $Q(2x)$ for an initial fortune of x. Suppose further that if he does reach the goal of $\frac{1}{2}$, or if he starts with fortune at least $\frac{1}{2}$ in the first place, he continues with ordinary bold play. For an initial fortune $x \ge \frac{1}{2}$, the overall chance of success is of course $Q(x)$, and for an initial fortune $x < \frac{1}{2}$, it is $Q(2x)Q(\frac{1}{2}) = pQ(2x) = Q(x)$. The success probability is indeed $Q(x)$ as for bold play, although the policy is different. With this example in mind, one can generate a whole series of distinct optimal policies.