

# 1 Direct solution to Poisson equation using Green's function

$$\nabla^2 D = f$$

The direct solution to the above Poisson equation is:

$$D(p) = \frac{1}{2} \int |x - p| f(x) dx$$

$$D(p, q) = \frac{1}{2\pi} \iint \log \left( \sqrt{(x - p)^2 + (y - q)^2} \right) f(x, y) dx dy$$

$$D(p, q, r) = \frac{-1}{4\pi} \iiint \frac{1}{\sqrt{(x - p)^2 + (y - q)^2 + (z - r)^2}} f(x, y, z) dx dy dz$$

That is, the Green functions in 1, 2 and 3 dimensions are

$$G(x, p) = \frac{1}{2} |x - p|$$

$$G(x, y; p, q) = \frac{1}{2\pi} \log \left( \sqrt{(x - p)^2 + (y - q)^2} \right)$$

$$G(x, y, z; p, q, r) = \frac{-1}{4\pi \sqrt{(x - p)^2 + (y - q)^2 + (z - r)^2}}$$

## 1-dimension

$$\nabla^2 D = 1 \text{ for } a < x < b.$$

$$D(p) = \frac{1}{2} \int |x - p| dx$$

$$= \frac{1}{2} \left\{ \int_a^p (p - x) dx + \int_p^b (x - p) dx \right\}$$

$$= \frac{1}{2} \left\{ \left[ px - \frac{x^2}{2} \right]_a^p + \left[ \frac{x^2}{2} - px \right]_p^b \right\}$$

$$= \frac{1}{2} \left\{ \left( p^2 - \frac{p^2}{2} \right) - \left( ap - \frac{a^2}{2} \right) + \left( \frac{b^2}{2} - bp \right) - \left( \frac{p^2}{2} - p^2 \right) \right\}$$

$$= \frac{1}{2} \left\{ p^2 - (a + b)p + \frac{a^2 + b^2}{2} \right\}$$

Thus, the solution is quadratic.

$$D(x) = \frac{x^2}{2} - \frac{a + b}{2} x + \frac{a^2 + b^2}{4}.$$

Clearly,  $\nabla \cdot \nabla D(x) = 1$ .

### 3-dimension

$$\nabla^2 D = 3.$$

$$D(p, q, s) = \frac{-1}{4\pi} \iiint \frac{1}{\sqrt{(x-p)^2 + (y-q)^2 + (z-s)^2}} \cdot 3 \, dx dy dz$$

Converting to spherical coordinates,

$$x = r \sin \theta \cos \phi + p$$

$$y = r \sin \theta \sin \phi + q$$

$$z = r \cos \theta + s$$

we get

$$r = \sqrt{(x-p)^2 + (y-q)^2 + (z-s)^2}$$

$$\phi = \tan^{-1} \left( \frac{y-q}{x-p} \right)$$

$$\theta = \cos^{-1} \left( \frac{z-s}{r} \right)$$

Then, the integral becomes

$$\begin{aligned} D(r, \theta, \phi) &= \frac{-1}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^a \frac{1}{r} \cdot 3 \, r^2 \sin \theta \, dr d\theta d\phi \\ &= \frac{-3}{4\pi} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^a r \sin \theta \, dr d\theta d\phi \\ &= \frac{-3}{4\pi} \left( \int_{\phi=0}^{2\pi} d\phi \right) \left( \int_{\theta=0}^{\pi} \sin \theta \, d\theta \right) \left( \int_{r=0}^a r \, dr \right) \\ &= \frac{-3}{4\pi} (2\pi) (-\cos \pi + \cos 0) \left( \frac{a^2}{2} \right) \\ &= \frac{-3}{4\pi} \cdot 2\pi \cdot 2 \cdot \frac{a^2}{2} \\ &= \frac{-3}{2} a^2 \end{aligned}$$

Converting back to Cartesian,

$$D(p, q, s) = \frac{-3}{2} (a_x - p)^2 + (a_y - q)^2 + (a_z - s)^2$$

Thus,  $D$  is quadratic in  $p, q, s$ . However,  $\nabla \cdot \nabla D = -9$ , but we tried to solve  $\nabla \cdot \nabla D = 3$ .  
What did go wrong ?