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Given an $f(x)$ that is increasing everywhere over $[a,b]$, take step functions $\Psi(x)$ and $\psi(x)$ such that

$$\Psi(x) = \begin{cases} f(x_1) & \text{if } x_0 \le x < x_1 \\ f(x_2) & \text{if } x_1 \le x < x_2 \\ \vdots \\ f(x_k) & \text{if } x_{k-1} \le x < x_k \end{cases} \qquad \text{and} \qquad \psi(x) = \begin{cases} f(x_0) & \text{if } x_0 \le x < x_1 \\ f(x_1) & \text{if } x_1 \le x < x_2 \\ \vdots \\ f(x_{k-1}) & \text{if } x_{k-1} \le x < x_k \end{cases}$$

where $[a,b]$ is divided into k equal parts, $x_0 = a$, and $x_k = b$.

Because $f(x)$ is everywhere-increasing, $\Psi(x) \ge f(x) \ge \psi(x)$. Choose k such that $\frac{b-a}{k}(f(b) - f(a)) < \epsilon$. Then,

$$\begin{aligned} \int_a^b \Psi(x)\mathrm{d}x - \int_a^b \psi(x)\mathrm{d}x &= \frac{b-a}{k}\left[\sum_{i=1}^{k} f(x_i) - \sum_{i=0}^{k-1} f(x_i)\right] \\ &= \frac{b-a}{k}\left[\left(\sum_{i=0}^{k} f(x_i) - f(x_0)\right) - \left(\sum_{i=0}^{k} f(x_i) - f(x_k)\right)\right] \\ &= \frac{b-a}{k}\left[f(x_k) - f(x_0)\right] \\ &= \frac{b-a}{k}\left[f(b) - f(a)\right] < \epsilon \end{aligned}$$