Proof that Cosmological redshift is equivalent to Doppler red shift

This proof is based on that presented by Jayant N Narlikar in 'Spectral shifts in General Relativity' in the American Journal of Physics October 1994. It corrects the invalidity of Narlikar's proof arising from his use of bases that do not exist, as well as an error in Equation [12] that generates errors in equations 13-17. 20 and 21. Equation numbers are chosen to match Narlikar's where applicable.

The FLRW metric centred at S gives the following formula for the line element:

$ds^2 = dt^2 - a(t)^2[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + sin^2\theta\ d\phi^2)]$ **[7]**

t, r, θ and ϕ are the 'comoving coordinates'. r is often denoted by χ, but here we use r. Where we do not explicitly indicate a coordinate system, the comoving coordinate system will be assumed. $a(t)$ is the cosmic scale parameter at cosmic time t. The values of the metric components in this coordinate system are:

$g_{00} = 1;\ g_{11} = -\frac{a(t)^2}{1-kr^2};\ g_{22} = -a(t)^2r^2;\ g_{33} = -a(t)^2r^2sin^2\theta$ **[7a]**

All other components (the off-diagonal ones) are zero. We will only need to use the first two of the four nonzero components. Since the metric is diagonal in the comoving coordinates, we also have the following inverse components:

$g_{00} = 1;\ g_{11} = -\frac{1-kr^2}{a(t)^2};\ g_{22}$ and g_{33} are not used. **[7b]**

Let S denote the spacetime event of emitting the signal that is seen by the observer at the spacetime event denoted by O.

Let $\vec{V}_S$ be the four-velocity of the source at S, which is stationary with respect to the CMBR. By definition $\|\vec{V}_S\| = 1$ **[9a]**.

We will be using three different reference frames in what follows:

- the FLRW frame centred on S (the "FS frame")
- the FLRW frame centred on O (the "FO frame")
- the inertial (Lorentz) frame of the observer at O (the "MO frame")

For coordinate-dependent items the applicable coordinate system will be denoted by a pre-subscript such as in ${}_{FO}W_S$, except that we will omit the pre-subscript for items expressed in the FS frame, as that is the one we will use the most.

Parameterise the null geodesic Λ from S to O by $\lambda : [0,1] \to \Lambda$.

Let $\bar{\vec{V}}_S$ be the result of parallel transporting $\vec{V}_S$ along the null geodesic Λ from S to O, and let $\vec{V}^*(u)$ be the intermediate transported vector at $\lambda(u)$.

Along Λ we have, from [7], since θ and ϕ do not change, and $ds = 0$: $\frac{dr}{\sqrt{1-kr^2}} = \frac{dt}{a(t)}$ **[10]**

The sign is positive because r increases with t (recall that r is radial distance from S).

First we need to write the equation for the geodesic Λ in terms of an affine parameter u. So the geodesic is $\lambda : [0,1] \to \Lambda$. To do this we write the following geodesic equation. But first, note that in what follows we do not specify the spatial origin of our FLRW spatial coordinates. Hence equations 11-16a are valid for any FLRW system, provided the value of r appropriate to the given system is used.

$\frac{d^2x^i}{du^2} + \Gamma^i{}_{kl}\ \frac{dx^k}{du}\ \frac{dx^l}{du} = 0$ [**11** - see Schutz 6.51]
We calculate the Christoffel symbol's value for i=0 as follows:

$$\begin{aligned}
\Gamma^0{}_{kl} &= \frac{1}{2}g^{0\beta}(g_{\beta k,l} + g_{\beta l,k} - g_{kl,\beta}) \quad \textbf{[11a - see Schutz 6.32]} \\
&= \frac{1}{2}g^{00}(g_{0k,l} + g_{0l,k} - g_{kl,0}) \quad \text{[since } g_{0\beta} = 0 \text{ unless } \beta = 0\text{]} \\
&= \frac{1}{2}g^{00}(g_{00,l} + g_{00,k} - g_{kl,0}) \\
&= -\frac{1}{2}g_{kl,0}\text{[since } g_{00} = g^{00} = 1\text{, which is constant]}
\end{aligned}$$

Hence [11] for i=0 becomes:

$$\begin{aligned}
0 &= \frac{d^2t}{du^2} - \frac{1}{2}g_{kl,0}\frac{dx^k}{du}\frac{dx^l}{du} \\
&= \frac{d^2t}{du^2} - \frac{1}{2}g_{00,0}(\frac{dt}{du})^2 - \frac{1}{2}g_{11,0}(\frac{dr}{du})^2 \text{ [since } \frac{d\theta}{du} \text{ and } \frac{d\phi}{du} \text{ must be zero]} \\
&= \frac{d^2t}{du^2} - \frac{1}{2}\frac{\partial(\frac{-a(t)^2}{1-kr^2})}{\partial t}(\frac{dr}{du})^2 \text{ [since } g_{00} \text{ is constant at 1]} \\
&= \frac{d^2t}{du^2} + (\frac{dr}{du})^2\frac{a(t)a'(t)}{1-kr^2} \quad \textbf{[12]} \\
&= \frac{d^2t}{du^2} + (\frac{dr}{dt})^2(\frac{dt}{du})^2\frac{a(t)a'(t)}{1-kr^2} \\
&= \frac{d^2t}{du^2} + \frac{1-kr^2}{a(t)^2}(\frac{dt}{du})^2\frac{a(t)a'(t)}{1-kr^2} \text{ [by 10]} \\
&= \frac{d^2t}{du^2} + (\frac{dt}{du})^2\frac{a'(t)}{a(t)} \\
&= \frac{1}{a(t)}\frac{d(a(t)\frac{dt}{du})}{du}
\end{aligned}$$

Hence, as $a(t) \neq 0$ we have $0 = \frac{d(a(t)\frac{dt}{du})}{du}$, whence:
$a(t)\frac{dt}{du} = A$ for some constant A. **[13]**
Hence:

$$\begin{aligned}
\frac{dr}{du} &= \frac{dr}{dt}\frac{dt}{du} \\
&= \frac{A}{a(t)}\frac{\sqrt{1-kr^2}}{a(t)} \text{ [by 13 and 10]} \\
&= \frac{A\sqrt{1-kr^2}}{a(t)^2} \quad \textbf{[14]}
\end{aligned}$$

This enables u to be determined as a function of r (as the light moves from S to O) and hence of t. It is convenient and permissible to set u=0 at S and u=1 at O. Integrating [13] we obtain:

$A = \int_0^1 A\ du = \int_{t(S)}^{t(O)} a(t)dt$ **[15]**
Let the tangent vector to the geodesic Λ at $\lambda(u)$ be $U(u)$. In the FS coordinates this has components $[\frac{dt}{du}, \frac{dr}{du}, \frac{d\theta}{du}, \frac{d\phi}{du}]$. The last two are zero and the first two are given by [13] and [14], hence the components are:
$U(u) = [\frac{A}{a(t)}, \frac{A\sqrt{1-kr^2}}{a(t)^2}, 0, 0]$ **[16a]**
and at O we have:
$U(1) = [\frac{A}{a_O}, \frac{A\sqrt{1-kr_O^2}}{a_O^2}, 0, 0]$ **[16b]**
where r_O is the radial comoving coordinate of O in the FS coordinate system. Equation 16b is the first equation that assumes a specific centre (S) for the FLRW system.

Let the vector $\vec{V}_S$ parallel transported from S to $\lambda(u)$ be $\vec{V}^*(u)$, and let us denote $\vec{V}^*(1)$ by $\vec{\bar{V}}_S$. Then, as parallel transport preserves magnitude and direction, both $||\vec{V}^*(u)|| = g(\vec{V}^*(u), \vec{V}^*(u))$ and $g(\vec{V}^*(u), \vec{U}(u))$ must be constant over u. **[17]**

Hence $||\vec{\bar{V}}_S|| = ||\vec{V}^*(1)|| = ||\vec{V}^*(0)|| = ||\vec{V}_S|| = 1$ [by 9a]. **[18a]**

And $g(\vec{\bar{V}}_S, \vec{U}(1)) = g(\vec{V}^*(1), \vec{U}(1)) = g(V^*(0), U(0)) = g(\vec{V}_S, \vec{U}(0))$ **[18b]**

Now, from the corollary in [ref1] we must have $V^{*2}(u) = V^{*3}(u) = 0$.

So, from [18a] we get:
$1 = \|\vec{\bar{V}}_S\| = g(O)(\vec{\bar{V}}_S, \vec{\bar{V}}_S) = g_{ik}(O)\bar{V}_S^i\bar{V}_S^k = g_{00}(O)(\bar{V}_S^0)^2 + g_{11}(O)(\bar{V}_S^0)^2$
[as off-diagonal elements of g are zero everywhere in spacetime [from 7] and $\bar{V}_S^2 = \bar{V}_S^3 = V^{*2}(1) = V^{*3}(1) = 0$]
$= (\bar{V}_S^0)^2 - \frac{a_O^2}{1-kr_O^2}(\bar{V}_S^1)^2$

Hence $(\bar{V}_S^0)^2 - \frac{a_O^2}{1-kr_O^2}(\bar{V}_S^1)^2 = 1$ **[19]**

From [18b] we get $g_O(\vec{\bar{V}}_S, \vec{U}(1)) = g_S(\vec{V}_S, \vec{U}(0))$
hence $g_{ik}(O)\bar{V}_S^i U^k(1) = {}_{FO}g_{ik}(S)_{FO}\ V_S^i{}_{FO}\ U^k(0)$ **[19a]**
Note that, since we are operating in T_SM for the right-hand side, we have to switch to FO, the FLRW basis centred at O, in order for the components to be well-defined.

The left-hand side of 19a is:
$= g_{00}(O)\bar{V}_S^0\ U^0(1) + g_{11}(O)\bar{V}_S^1\ U^1(1)$ [since $\bar{V}_S^2 = \bar{V}_S^3 = 0$]
$= \bar{V}_S^0\ U^0(1) - \frac{a_O^2}{1-kr_O^2}\bar{V}_S^1\ U^1(1) = \bar{V}_S^0\frac{A}{a_O} - \frac{a_O^2}{1-kr_O^2}\frac{A\sqrt{1-kr_O^2}}{a_O^2}\bar{V}_S^1$ [by 16b]
$= A(\frac{\bar{V}_S^0}{a_O} - \frac{\bar{V}_S^1}{\sqrt{1-kr_O^2}})$

The right-hand side of 19a is:

$= {}_{FO}g_{00}(S)_{FO}V_S^0{}_{FO}U(0)^0 + {}_{FO}g_{11}(S)_{FO}V_S^1{}_{FO}U(0)^1$

[since $U(0)^2 = U(0)^3 = 0$ because the light ray is radial (again from [ref1]).
$= {}_{FO}g_{00}(S)_{FO}U(0)^0$

[since the FLRW spatial coordinates of S are constant, so $\vec{V}_S = [1, 0, 0, 0]$ in the FO frame]
$= {}_{FO}U(0)^0$ [by 7a] $= \frac{A}{a_S}$ [by 16a]

Hence, equating the right and left sides of 19a we get:

$\frac{\bar{V}_S^0}{a_O} - \frac{\bar{V}_S^1}{\sqrt{1-kr_O^2}} = \frac{1}{a_S}$ **[20]**

Next we note that the four-velocity of the observer has components $[1,0,0,0]$ in any FLRW frame (because the observer has zero spatial coordinate velocity in that frame) and also in the observer's *inertial* frame at O. Hence the time basis vectors of the MO and FS frames must be identical: $\vec{e}_0(O) = {}_{MO}\vec{e}_0$.

Now $\vec{\bar{V}}_S = {}_{MO}\bar{V}_S^i \; {}_{MO}\vec{e}_i = \bar{V}_S^i \; \vec{e}_i(O) = \bar{V}_S^0 \; \vec{e}_0(O) + \bar{V}_S^1 \vec{e}_1(O)$
$= \bar{V}_S^0 \; {}_{MO}\vec{e}_0 + \bar{V}_S^1 \; \vec{e}_1(O)$

Hence, since both bases are orthogonal, we have ${}_{MO}\bar{V}_S^0 = \bar{V}_S^0$ and, by rotating the Lorentz frame appropriately around O, we can without loss of generality choose our basis vector ${}_{MO}\vec{e}_1$ so that it aligns with $\vec{e}_1(O)$ but with opposite direction (ie ${}_{MO}\vec{e}_1$ points along the radial line from O to S while $\vec{e}_1(O)$ points from O in the opposite direction from S). Then, since $\vec{\bar{V}}_S = [\bar{V}_S^0, \bar{V}_S^1, 0, 0]$ in the FS basis, we can write $\vec{\bar{V}}_S = [\gamma, \gamma V, 0, 0]$ in the MO basis, where ${}_{MO}\bar{V}_S^0 = \gamma = \frac{1}{\sqrt{1-V^2}} = \bar{V}_S^0$ and V has the opposite sign to $\bar{V}_S^1$ (because γ is positive and the corresponding basis vectors, ${}_{MO}\vec{e}_1$ and $\vec{e}_1(O)$, point in opposite directions).

Hence $||\vec{\bar{V}}_S|| = g(\vec{\bar{V}}_S, \vec{\bar{V}}_S) = \gamma^2 - (\gamma V)^2 = \gamma^2 - \frac{a_O^2}{1-kr_O^2}(\bar{V}_S^1)^2$ where we calculate the magnitude in the MO and FS bases and equate the results.
Hence $(\gamma V)^2 = \frac{a_O^2}{1-kr_O^2}(\bar{V}_S^1)^2$ and so
$\gamma V = -\frac{a_O}{\sqrt{1-kr_O^2}}\bar{V}_S^1$ **[21]**
where the sign is negative because γ is positive and V and $\bar{V}_S^1$ have opposing signs.

Now the red shift is given by:
$1 + z = \frac{a_O}{a_S}$ **[8]**
$= a_O\left(\frac{\bar{V}_S^0}{a_O} - \frac{\bar{V}_S^1}{\sqrt{1-kr_O^2}}\right)$ [by 20]
$= \gamma - \frac{a_O}{\sqrt{1-kr_O^2}} \; \bar{V}_S^1 = \gamma + \gamma V$ **[by 21]**
$= \gamma(1+V) = \frac{1+V}{\sqrt{1-V^2}} = \sqrt{\frac{(1+V)^2}{(1-V)(1+V)}}$
$= \sqrt{\frac{1+V}{1-V}}$ **[22]**
This is the formula for a Doppler shift within a Lorentz frame, for light arriving from an object that is receding with radial velocity V.

ref1: Post 4 of http://www.physicsforums.com/showthread.php?t=624117