**Calculus, the Elements, Michael Comenetz.**

This book is unlike any other calculus book I have seen. It is an extremely scholarly work, by a pure mathematician with research credentials, as well as years of teaching experience, whose goal seems to be to explain the ideas behind the calculus, as well as its origins and applications, to the intelligent and curious but mathematically unsophisticated beginner, even lay persons. Repeatedly, in reading it, I have found cogent explanations of observations which are omitted in other books, and which I have myself formed a habit of adding to the courses I teach. But there are also many explanations I have never thought of including in my courses.

Often in calculus courses, it is tempting to present the material in its purely mathematical form, into which it has evolved over several centuries. To reduce prerequisites, we delete many applications to physics, leaving only a few illustrations in geometry. The reciprocal relationship between derivative and integral is usually "established" at least theoretically, by proving the fundamental theorem of calculus, which students unfortunately remember only as a computational device for computing areas. Of the two parts of the FTC, the first part is dismissed and largely misunderstood by the average student, and the second part is memorized as a rule for calculating.

Professor Comenetz takes a different approach, aimed at conveying first an intuitive grasp of the ideas calculus is designed to capture and measure, before presenting their mathematical formulation. Once these ideas have been made familiar through several well chosen examples, he gradually makes the concepts more precise and explains how they are rendered into mathematics, i.e. how physical quantities become numbers, and finally how they may be computed.

There are many possible goals one can attempt in teaching. E.g.

1) How does one carry out computations?
2) Why are the computational procedures correct?
3) What are the computations good for?
4) How did people think of these procedures, or how could one do so by analyzing examples?
5) How can one acquire an intuitive "feel" for the methods?

It is increasingly common, and easier, to address only question 1, or 1 and 3, i.e. to teach only for "competency" as opposed to understanding. It is striking that in this book, with only about 400 pages of text, and not too heavy to carry around, the author has made an honest attempt to do all of the above. This book is far from a cookbook for the below average student. It is a scholarly, meticulously written explanation of calculus for the intelligent person who wants to understand the subject.

This text differs radically from other calculus books I have seen. In contrast to the depressing trend of books titled something like "Calculus for dummies", whose goal is to program people to perform computations they do not understand, this book makes a serious and well thought out attempt to thoroughly explain the ideas of calculus. Even the most successful students of a standard calculus course come away thinking perhaps that a derivative measures slope, and an integral measures area. Professor Comenetz makes it absolutely clear with many examples that integral and derivative are not specific quantities, but relationships between pairs of quantities. I.e. there are many pairs such that the first is the integral of the second, and equivalently the second is the derivative of the first, such as (area, height) and (rise, slope), but also (mass, density), (increase in volume, rate of flow), (momentum, force), and others. As a mathematician with over thirty years research and teaching experience, but little grasp of physics, I had not realized the simple nature of some of these pairings before reading this book. And it is not everyday that I learn something from an introductory calculus book.

After reading this text, from an author who has taken his thorough understanding of the concepts involved, deepened and refined them by a scholarly investigation of their origins, and allowed them to mature over decades of teaching at a school where excellent teaching is of paramount importance, I will not be able to ignore the message of it, even in courses where it is not the book of choice.

I have tried for years to teach my students to understand and remember the statements of the fundamental theorem of calculus, realizing slowly that the primary hindrance is the mathematical notation and formulation of the concepts. This book takes the now seemingly obvious logical step of teaching the concepts by example, before introducing the notation, and only then to render it more precise, and more abstract. I.e. it is hard to argue with the philosophy of this book, to teach a concept well, one should first teach the ideas, as they actually arose, and only afterward should one abstract them and render them into mathematical language. Indeed this language was invented as a means of rendering into mathematics, physical ideas which have all too often been omitted from current calculus courses, as too time consuming to discuss.

In spite of taking an example oriented approach to the introduction to calculus, this book still presents a perfectly rigorous, i.e. correct and precise, account of the logical development of the mathematical part of the subject. I.e. proofs are given, and they are clearly written, and are logically correct in every respect. This book is thus not only more intuitive in its approach to calculus, but also more logically rigorous in its discussion of the theoretical side than is usual. That is not to say it is exhaustive. The approach here to the theory of calculus parallels that taken to the intuition behind calculus. I.e. it is intended not only that the proofs should be valid , but that one should understand them. Two features make this more likely in the present book than in many others. First of all the arguments are scrupulously correct, and presented by someone who clearly not only understands them thoroughly, but has even researched their origins in the history of mathematical writing.

For the lay person let me say that it is common in mathematics texts for authors to continue to present arguments which have become traditional in the most popular books, for various reasons, even if superior arguments were in existence in earlier works. This does not occur in the present book. Professor Comenetz often presents, instead of the most common recent argument for a given result, either one of his own creation, or the most insightful one to be found anywhere in the literature, and his familiarity with that literature extends to Isaac Newton and Euclid. As an example, the existence of the integral of a continuous function, is the result usually stated but not proved in most beginning texts. Some will even present a proof but one offering little useful information to a beginner. Instead this book contains a clear and elementary proof for the existence of the integral of a monotone function, which I learned from the excellent text of Apostol. Professor Comenetz gives it clearly and succinctly and then cites to a reference for it in the Principia of Isaac Newton which I have never been aware of before.

After making clear with this elementary argument the essential point of an existence proof for integrals, he later introduces the concept of uniform continuity, which is needed for the general argument involving continuous functions. Instead of sweeping aside entirely the argument that all continuous functions on a closed bounded interval are in fact uniformly continuous, he observes that from what has already been explained, the result is easy for continuously differentiable functions, a class which comprises all examples any standard course will treat.

This approach is taken throughout. I.e. when a theorem has a difficult proof, instead of brutally presenting it, or callously omitting it, Professor Comenetz illustrates the key idea by giving a proof of an easier case, and then points out how to enhance the argument to reach the more difficult case. This style of explanation is well chosen to guide the serious beginner.

In reference to the Fundamental theorem of calculus, even the statement surprised me. Although in essence the statement is equivalent to the usual one, it has been carefully and intelligently phrased so as to reveal more perfectly than I have ever seen, the perfect duality between the integral and derivative. This is indeed an unusually thoughtful and scholarly book, especially for one aimed at the beginner. A course based on it would in my opinion definitely have a much greater chance of producing students who understand the structure, uses, and arguments of calculus, than is usually the case.

Another unusual feature of this book is its attempt to incorporate the concept of "infinitesimals" into the discussion. In reading these sections I admit to feeling some kinship with Bishop Berkeley who complained of his inability to understand Newton. After almost 40 years of personal mathematical atheism, the author almost persuades me to believe in infinitesimals, and he certainly makes me want to. Although I have always preferred the simplicity of the pure mathematical formulation of calculus, after reading this book it is hard to be satisfied with only that. There seems no doubt that the long experience of the author in teaching at a school (St. Johns College) where the reading of original texts is a hallowed institution, has very beneficially influenced his preparation for this work.

As to pedagogy, consider the advantage of thinking of the integral and derivative as illustrating concepts the student already believes to be inverse, as opposed to trying to force on him the reciprocal nature of two ideas which are both defined only mathematically. Thus after taking the time to illustrate that the density of a wire at a point is approximated by its average densities over nearby intervals, and reciprocally that the mass of a small segment of wire where the density varies only slightly, is approximated by that of a wire of uniform density, it becomes almost trivial to believe that the integral and derivative (abstract versions of mass and density) determine each other in the same way. In the more standard example, area under a graph over a short interval where the height varies only slightly, is approximated by that of a graph of constant height, and reciprocally the height of a graph at a point is approximated by the quotient of the area taken over a small interval containing that point, divided by the width of that interval.

These are examples of the author's philosophy that a mathematical definition should not be attempted until the concept involved has been sufficiently explained so that the reader can well judge whether the abstract definition presented does indeed capture the idea which is to be made precise.

A few more specific things I like in the book:

1. The frank explanation that the primary way to evaluate limits is to reduce to the case of evaluating a continuous function at a point. (Although this is the primary method used in standard books, few if any acknowledge it clearly.)

2. The clear explanation of the special role of endpoints in defining continuity and differentiability.

3. The inclusion of l'Hopital's rule, and in a case where its proof is trivial, immediately after introducing derivatives. (This indispensable tool for evaluating limits is as of next year omitted from our own second semester course in calculus.)

4. The clear explanation of the meaning of the symbol delta x, especially the (lack of) condsequences of it taking negative values.

Although it contains relatively few routine computational problems, these are easily added from a workbook like Schaum's outline series. I actually prefer a book which offers a small number of thoughtful problems which the student should work all of, instead of a huge number which can (and will) mostly be ignored.

If we mathematicians would begin again to ask ourselves about the origins, applications, and meaning, of our symbols, we may regain the interest and appreciation of the public, of our colleagues in other disciplines, and of our students. This book is a good start in the right direction. I recommend this work to high school and college teachers looking for a book from which they and their students are likely to learn much about calculus and about pedagogy, as well as to the scientifically curious layman.

There are certainly other excellent and classic calculus books available (such as those by Courant, or Apostol, or even Sylvanus P. Thompson), but in my opinion many recent and popular works on the topic will appear intellectually sterile after exposure to this one. The only challenge which the author has not attempted is that of trying to explain the calculus to readers who do not read English well. Virtually every sentence in this book says exactly what the author intended it to, but it may require some reflection. This is no doubt intended. Wherever there are students for whom phrases like "rectilinear motion" do not require to be rendered into words of at most 2 syllables, as "motion along a straight line", this book will be very useful and stimulating indeed.

Roy Smith
Professor of mathematics
University of Georgia, Athens, GA