PHYS-2010: General Physics I
Course Lecture Notes
Section VIII

Dr. Donald G. Luttermoser
East Tennessee State University

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Abstract

These class notes are designed for use of the instructor and students of the course PHYS-2010: General Physics I taught by Dr. Donald Luttermoser at East Tennessee State University. These notes make reference to the *College Physics, 7th Edition* (2005) textbook by Serway and Faughn.

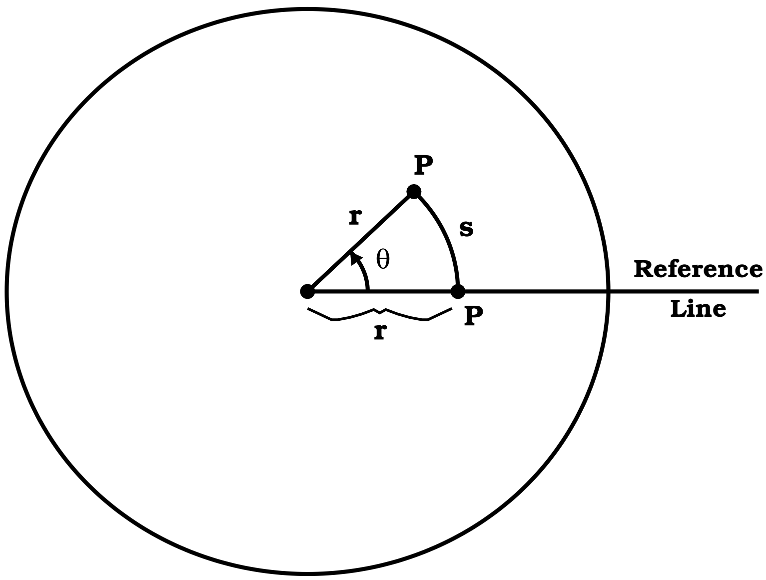
VIII. Circular Motion

A. Angular Speed and Angular Acceleration.

1. **Arc length** of a rotating or revolving object is analogous to linear displacement:

   a) Arc length, s, is swept out as an object rotates θ degrees such that

$$\boxed{s = \theta\, r \ .} \qquad \text{(VIII-1)}$$



i) s is measured in the same units as r (*i.e.*, a length).

ii) θ is measured in **radians** (not degrees).

$$\boxed{1 \text{ revolution (rev)} \ = 2\pi \text{ radians (rad)} \ = 360^\circ \ ,} \qquad \text{(VIII-2)}$$

or

$$1 \text{ rad} \ \equiv \frac{360^\circ}{2\pi} = 57.^\circ 3 \ .$$

b) Conversion of angles ($\theta \equiv$ **angular displacement**):

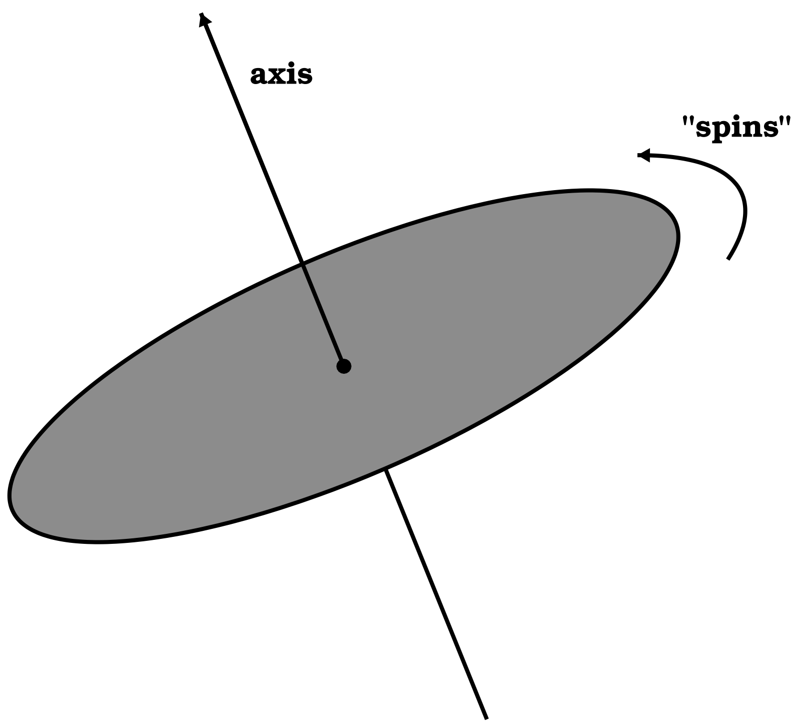
$$\boxed{\theta \text{ (rad)} = \frac{\pi}{180^\circ}\, \theta \text{ (deg)}\ ,} \qquad \text{(VIII-3)}$$

$$\theta \text{ (rad)} = 1.745 \times 10^{-2}\, \theta \text{ (deg)}\ .$$

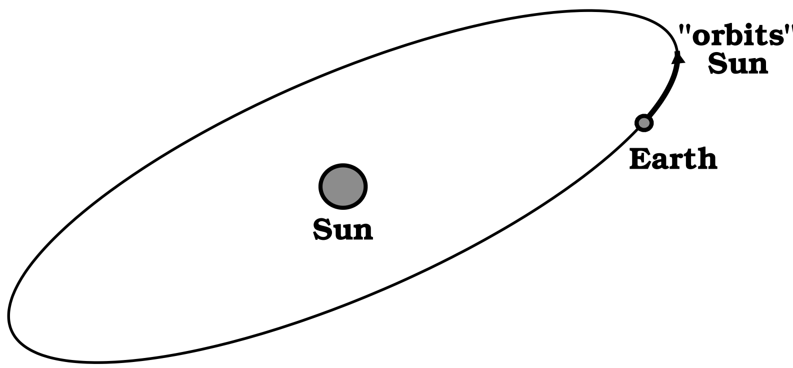
c) Sometimes you may run across angular measure dealing with very small angles. There are 60' (arcminutes = arcmin) per 1° and 60" (arcseconds = arcsec) per 1 arcminute:

$$1^\circ = 60' = 3600''$$

d) An object is said to be **rotating** if it is spinning about an axis (*e.g.*, the Earth "rotates" about its axis).



e) A body **revolves** around another object if the first body is in **orbit** about the second object (*e.g.*, the Earth "revolves" about the Sun).



Example VIII–1. **Problem 7.2 (Page 218) from the Serway & Faughn textbook**: A wheel of radius 4.1 meters rotates at a constant velocity. How far (path length) does a point on the circumference travel if the wheel is rotated through angles of 30°, 30 rad, and 30 rev?

Let r be the radius of the wheel, s be the path length along the circumference, and θ be the angle that this point subtends from a reference point as the wheel rotates. As such, $r = 4.1$ m, and $\theta_1 = 30°$, $\theta_2 = 30$ rad, and $\theta_3 = 30$ rev. We will use Eq. (VI-1),

$$s = \theta\, r \ ,$$

except that all of these angles must be expressed in radians, so

$$\begin{aligned}
\theta_1 &= 30° \times \frac{\pi \text{ rad}}{180°} = 0.5236 \text{ rad}, \\
\theta_2 &= 30 \text{ rad} \qquad \text{(which is what we want)}, \\
\theta_3 &= 30 \text{ rev } \times \frac{2\pi \text{ rad}}{1 \text{ rev}} = 188.5 \text{ rad}.
\end{aligned}$$

Plugging these values of θ into Eq. (VI-1) gives

$$s_1 = \theta_1 r = 0.5236\,(4.1 \text{ m}) = \boxed{2.1 \text{ m}}$$

$$s_2 = \theta_2 r = 30\,(4.1 \text{ m}) = \boxed{120 \text{ m}}$$

$$s_3 = \theta_3 r = 188.5\,(4.1 \text{ m}) = \boxed{770 \text{ m}}$$

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2. The length s in Eq. (VIII-1) is actually an arc length (curved length). However, if θ is small (*i.e.*, $\theta \ll 1$), $s \rightarrow D$, the actual linear size (or diameter if the object is round).

a) As such, by measuring the angle that a distance object subtends, we can calculate its actual size (or diameter) D by knowing the distance d to the object.

b) Since θ is small here, it is often more convenient to express this angle in terms of arcseconds instead of radians. We can rewrite Eq. (VIII-1) by setting $s = D$, $r = d$, and expressing θ in arcseconds and relabeling it as α, then

$$\begin{aligned} \theta(\text{rad}) &= \alpha(\text{arcseconds}) \times \frac{\pi \text{ rad}}{180^\circ} \times \frac{1^\circ}{3600 \text{ arcsec}} \\ &= \frac{\alpha \text{ arcsec}}{206,265 \text{ arcsec/rad}} \,. \end{aligned}$$

c) Using α instead of θ in Eq. (VIII-1) gives

$$D = \frac{\alpha\, d}{206,265} \,, \qquad \text{(VIII-4)}$$

which is called the **small-angle formula**. Here, D is the linear size (*i.e.*, diameter) of the object at a distance d which subtends an angle α measured in arcseconds (the

206,265 is the conversion factor between arcseconds and radians). The lengths d and D will be in the same units (*e.g.*, if d is measured in km, then D will be in km). Note that this formula is only valid when $\theta \ll 1$ (typically, one would want $\alpha < 1000$ arcseconds in order to use this small-angle approximation).

Example VIII–2. The star Betelguese (α Ori) has an angular size of 0.040 arcsec (40 milliarcseconds = 40 mas) and it is at a distance of 200 pc. What is the linear size of Betelgeuse? How does this size compare to the planet's distances from the Sun in our solar system?

$$d = 200 \text{ pc x } 3.09 \times 10^{16} \text{ m/pc} = 6.18 \times 10^{18} \text{ m}$$
$$D = 0.040 \text{ x } 6.18 \times 10^{18} \text{ m / } 206{,}265 = 1.20 \times 10^{12} \text{ m.}$$

The Sun is 1.39×10^9 m in diameter which means that Betelgeuse is

$$D = \frac{1.20 \times 10^{12} \text{ m}}{1.39 \times 10^{9} \text{ m/D}_\odot} = 860 \text{ D}_\odot = 8.0 \text{ AU},$$

where an Astronomical Unit (AU) is the size of the Earth's orbital semimajor axis (note that 1 AU = 1.496×10^{11} m). Betelgeuse is 860 times bigger than the Sun! This diameter gives a stellar radius of 4.0 AU for Betelgeuse. If it was put in the place of the Sun, then Mercury (0.39 AU), Venus (0.72 AU), Earth (1.0 AU), and Mars (1.5 AU) would be inside Betelgeuse! The planet Jupiter (5.2 AU) would be close (1.2 AU) from the photosphere ("surface") of Betelgeuse.

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3. **Angular speed**, ω, is the change of angular displacement divided by the time interval that the angular displacement took place.

a) Average angular speed:

$$\boxed{\overline{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}\ .} \qquad \text{(VIII-5)}$$

b) Instantaneous angular speed:

$$\boxed{\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}\ .} \qquad \text{(VIII-6)}$$

c) Angular speed has units of radians/second $\Longrightarrow$ rad/s (or just s^{-1} since "radian" is a pseudo-unit).

d) ω is positive when θ is increasing (counter-clockwise motion) and negative in the clockwise direction.

4. **Angular acceleration**, α (alpha), is the change of angular speed over the time interval of interest:

$$\boxed{\overline{\alpha} \equiv \frac{\text{change in angular speed}}{\text{time interval}} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}\ ,} \qquad \text{(VIII-7)}$$

which is the average angular acceleration.

a) Whereas the instantaneous acceleration is

$$\boxed{\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}\ .} \qquad \text{(VIII-8)}$$

b) The units of angular acceleration are rad/s^2 (or just s^{-2}).

5. When a rigid object rotates about an axis, every portion of the object has the same angular speed and same angular acceleration.

6. Note that we have <u>not</u> written then angular variables as vectors. In reality they are vectors, but the unit vectors for the magnitudes we have written above, point out of the plane in which the object is rotating. Since this is a bit complicated, we will just discuss the magnitudes of these angular motions at present.

B. Rotation Motion under Constant Angular Acceleration.

1. These are analogous to their counterparts in linear motion.

LINEAR	**ANGULAR**	
$v = v_\circ + at$	$\boxed{\omega = \omega_\circ + \alpha t \ ,}$	(VIII-9)
$x = v_\circ t + \frac{1}{2}at^2$	$\boxed{\theta = \omega_\circ t + \frac{1}{2}\alpha t^2 \ ,}$	(VIII-10)
$v^2 = v_\circ^2 + 2ax$	$\boxed{\omega^2 = \omega_\circ^2 + 2\alpha\theta \ .}$	(VIII-11)

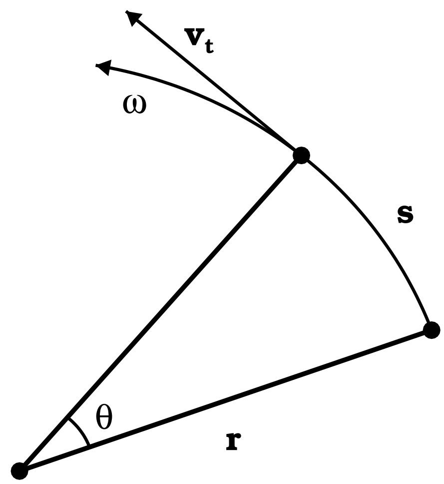
Note that in the above equations, the initial times, angles, and displacements are all set to zero. If these initial setting are not zero, replace t with Δt, θ with $\Delta\theta$, and x with Δx.

2. The **tangential speed** of a point on a rotating object equals the distance of that point from the axis of rotation (r) multiplied by the angular speed (ω):

$$\begin{aligned}
\Delta\theta &= \frac{\Delta s}{r} \ , \\
\underbrace{\frac{\Delta\theta}{\Delta t}}_{\overline{\omega}} &= \frac{1}{r}\underbrace{\frac{\Delta s}{\Delta t}}_{\overline{v}} \ , \\
\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t} &= \frac{1}{r}\lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} \ , \\
\omega &= \frac{1}{r}\, v \ ,
\end{aligned}$$

$$\boxed{v_{\rm t} = r\,\omega \ ,} \qquad \text{(VIII-12)}$$

where v_t is the "tangential" velocity <u>tangent</u> to the curved path of motion the particle is taking.



3. The **tangential acceleration**, a_t, of a point on a rotating object equals the distance of that point from the axis of rotation (r) multiplied by the angular acceleration (α):

$$\begin{aligned} \Delta v_t &= r\,\Delta\omega \ , \\ \underbrace{\frac{\Delta v_t}{\Delta t}}_{\overline{a}} &= r\,\underbrace{\frac{\Delta\omega}{\Delta t}}_{\overline{\alpha}} \\ \lim_{\Delta t\to 0}\frac{\Delta v_t}{\Delta t} &= r\,\lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} \ , \end{aligned}$$

$$\boxed{a_t = r\,\alpha\ .} \qquad \text{(VIII-13)}$$

C. Centripetal Acceleration and Force.

1. If an object travels at constant speed in a curved path, it <u>is</u> accelerating since the velocity vector is continuously changing direction.

2. Besides, the tangential acceleration (*i.e.*, acceleration tangent to the curved path), there must be an acceleration perpendicular ($\perp$) to the tangent line pointing toward the center of the curved path.

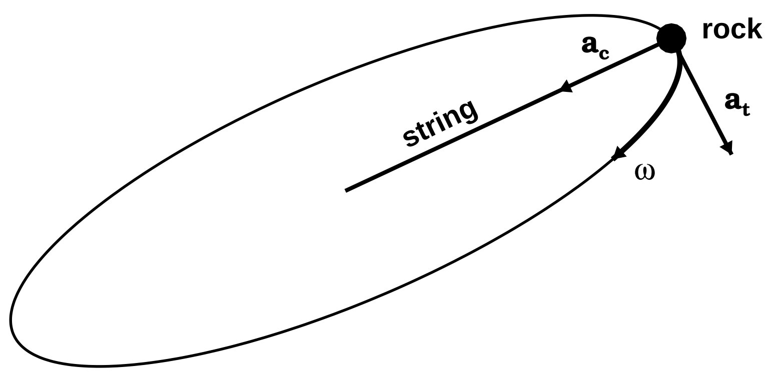
a) Otherwise, the object would fly off in a straight path in the direction of $a_{\rm t}$ as per Newton's 1st law of motion.

b) This *center-seeking* acceleration is called the **centripetal acceleration**,

$$\boxed{a_{\rm c} = r\,\omega^2 = \frac{v_{\rm t}^2}{r}\ .} \qquad \text{(VIII-14)}$$

i) Gravity is a centripetal force (hence acceleration) for planets orbiting the Sun.

ii) A string's tension is a centripetal force (hence acceleration) for an object attached to that string being rotated in circular motion.



c) The total acceleration of an object in circular motion is

$$\boxed{\vec{a} = \vec{a}_{\rm t} + \vec{a}_{\rm c}\ ,} \qquad \text{(VIII-15)}$$

$$\boxed{a = \sqrt{a_{\rm t}^2 + a_{\rm c}^2}\ .} \qquad \text{(VIII-16)}$$

Example VIII–3. Problem 7.18 (Page 219) from the Serway & Faughn textbook: A race car starts from rest on a circular track of radius 400 m. The car's speed increases at a constant rate of 0.500 m/s^2. At the point where the magnitudes of the centripetal and tangential accelerations are equal, determine (a) the speed of the race car, (b) the distance traveled, and (c) the elapsed time.

Solution (a):

From Eq. (VIII-14), the centripetal acceleration is $a_{\rm c} = \frac{v_{\rm t}^2}{r}$. Thus, when $a_{\rm c} = a_{\rm t} = 0.500$ m/s^2, we have

$$v_{\rm t} = \sqrt{r\,a_{\rm c}} = \sqrt{(400\ {\rm m})(0.500\ {\rm m/s^2})} = \boxed{14.1\ {\rm m/s}\ .}$$

Solution (b):

The total distance traveled is just the total arc length traveled on the circular track, s, which can be found from the linear 1-D equation of motion:

$$v^2 = v_\circ^2 + 2\,a\,s\ ,$$

where $v = v_{\rm t} = 14.1$ m/s, $v_\circ = 0$ (starts from rest), and $a = a_{\rm t} = 0.500$ m/s^2. Hence, the total arc length (*i.e.*, distance) traveled is

$$s = \frac{v^2 - v_\circ^2}{2\,a_{\rm t}} = \frac{(14.1\ {\rm m/s})^2 - 0}{2 \cdot 0.500\ {\rm m/s^2}} = \boxed{20\underline{0}\ {\rm m}\ .}$$

Solution (c):

We can use another linear 1-D equation of motion to find the elapse time:

$$v = v_\circ + a\,t\ ,$$

where $v = v_{\rm t} = 14.1$ m/s, $v_\circ = 0$ (starts from rest), and $a = a_{\rm t} = 0.500$ m/s^2. Hence, the time it take to reach the tangential velocity from part (a) is

$$t = \frac{v_t - v_i}{a_t} = \frac{14.1\ \text{m/s} - 0}{0.500\ \text{m/s}^2} = \boxed{28.2\ \text{s}}\ .$$

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3. The centripetal force is just the mass times the centripetal acceleration:

$$\boxed{F_c = m\,a_c = \frac{m\,v_t^2}{r} = m\,r\,\omega^2\ .} \qquad \text{(VIII-17)}$$

a) Centripetal forces are "center-seeking" forces.

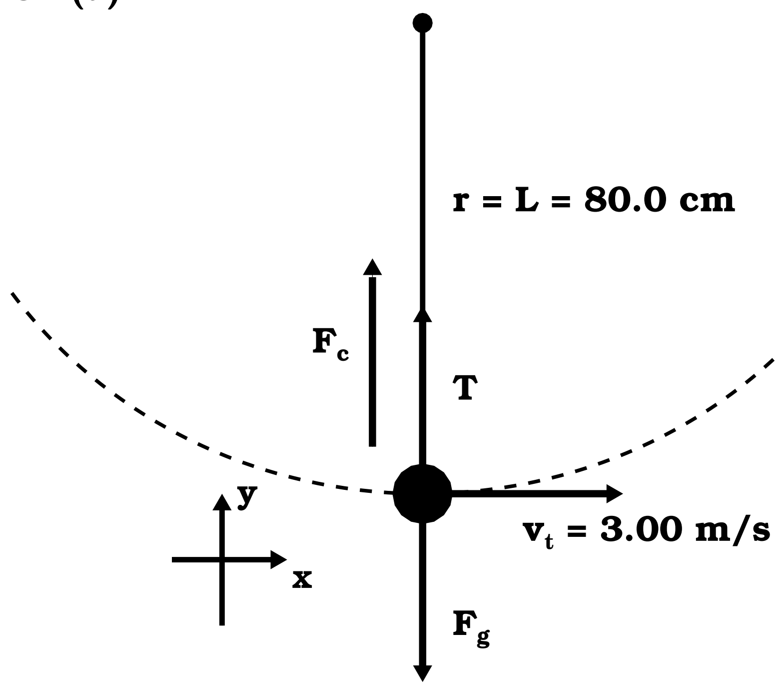
i) The gravitational force is a center-seeking force $\Longrightarrow$ planets are in orbit about the Sun.

ii) The tension force is a centripetal force for an object connected to a string being swung about some axis.

b) The concept of **centrifugal force**, the apparent force that forces an object outward while moving in a circular path, is not really a force $\Longrightarrow$ this tendency to follow a straight line path away from the curved path is nothing more than Newton's 1st law — the law of inertia.

Example VIII–4. Problem 7.48 (Page 221) from the Serway & Faughn textbook: A 0.400-kg pendulum bob passes through the lowest part of its path at a speed of 3.00 m/s. (a) What is the tension in the pendulum cable at this point if the pendulum is 80.0 cm long? (b) When the pendulum reaches its highest point, what angle does the cable make with the vertical? (c) What is the tension in the pendulum cable when the pendulum reaches its highest point?

Solution (a):



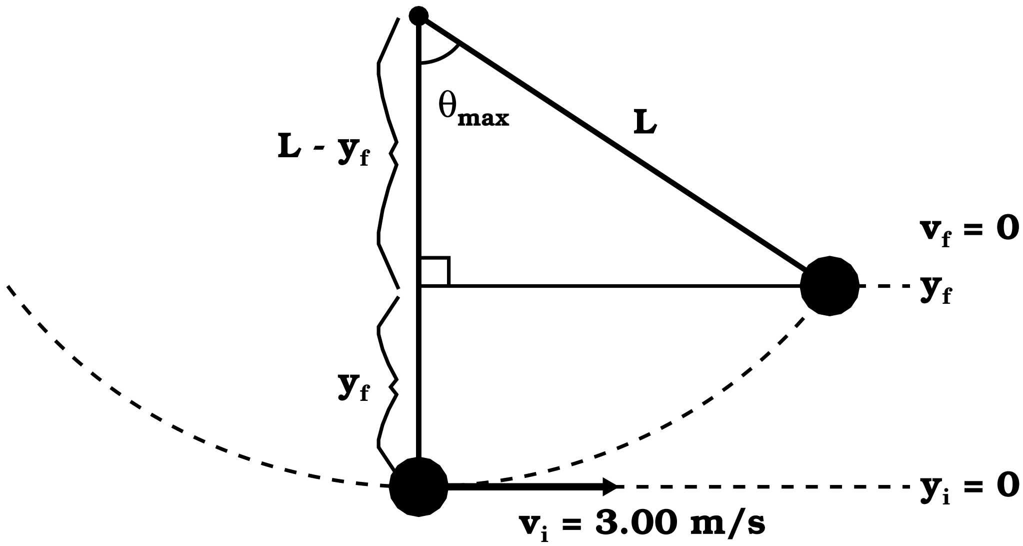
The diagram above shows the system when the bob is at its lowest point and shows the direction of the forces involved and the tangential velocity vector. The mass of the bob is $m = 0.400$ kg, and the length of the cable as $L = 80.0$ cm $= 0.800$ m. Since the bob follows a circular path about the pivot of the pendulum, the radius of this path is just the length of the cable, $r = 0.800$ m. The centripetal force is always directed toward the center of the "orbit," which we will define as the positive y direction as indicated in the figure above. As a result, the sum of the forces in the y direction is just the total centripetal force:

$$\sum F_y = F_{\rm c} = T - F_{\rm g} \ .$$

Solving for the tension T gives

$$\begin{aligned} T &= F_{\rm g} + F_{\rm c} = mg + ma_{\rm c} = m\,(g + a_{\rm c}) \\ &= m\left(g + \frac{v_{\rm t}^2}{r}\right) = (0.400 \text{ kg})\left(9.80 \text{ m/s}^2 + \frac{(3.00 \text{ m/s})^2}{0.800 \text{ m}}\right) \\ &= \boxed{8.42 \text{ N}} \ . \end{aligned}$$

Solution (b):



The diagram above shows the bob at its maximum height. To find this height, hence $\theta_{\rm max}$, we only need to use the conservation of mechanical energy. We will set the lowest position of the bob at the "ground" level, $y_{\rm i} = 0$. At its highest position, $y_{\rm f}$, the velocity changes from the counter-clockwise to the clockwise direction, hence goes through zero at this point, $v_{\rm f} = 0$. From the diagram above, we can solve for $y_{\rm f}$ in terms of $\theta_{\rm max}$:

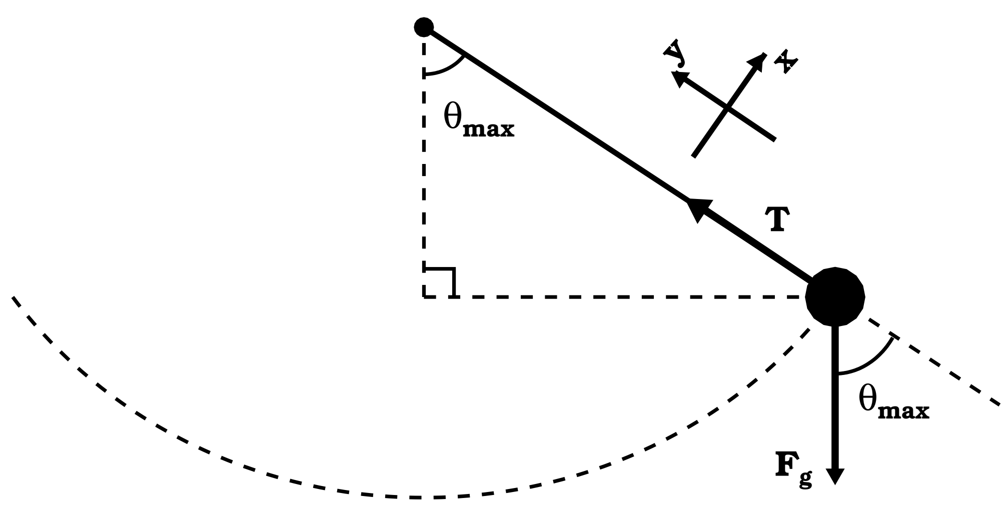
$$
\begin{aligned}
\cos\theta_{\rm max} &= \frac{L - y_{\rm f}}{L} \\
L\,\cos\theta_{\rm max} &= L - y_{\rm f} \\
y_{\rm f} &= L - L\,\cos\theta_{\rm max} \\
y_{\rm f} &= L\,(1 - \cos\theta_{\rm max})\ .
\end{aligned}
$$

We can now determine $\theta_{\rm max}$ from the conservation of mechanical energy by setting the lowest point as the initial position, so $v_{\rm i} = v_{\rm t}$ [from part (a)], and $y_{\rm i} = 0$, and the highest point as the final position, so $v_{\rm f} = 0$ and $y_{\rm f}$ is given in the equation above:

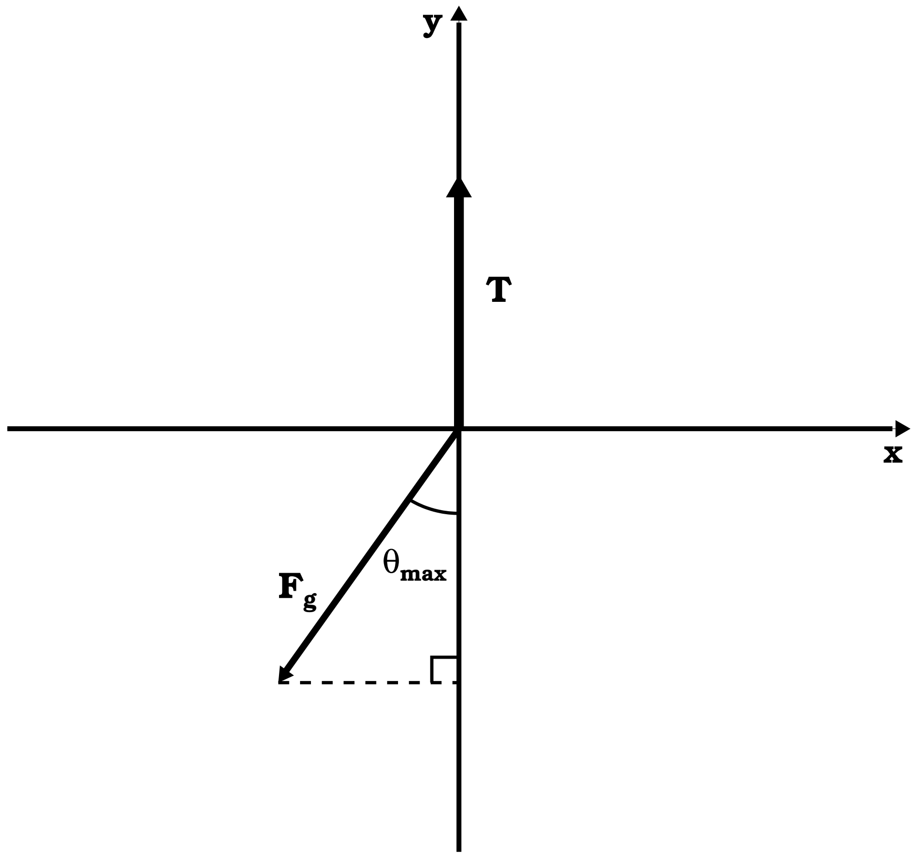
$$
\begin{aligned}
({\rm KE} + {\rm PE_g})_{\rm i} &= ({\rm KE} + {\rm PE_g})_{\rm f} \\
\frac{1}{2}mv_{\rm i}^2 + mgy_{\rm i} &= \frac{1}{2}mv_{\rm f}^2 + mgy_{\rm f}
\end{aligned}
$$

$$\begin{aligned}
\frac{1}{2}mv_{\rm t}^2 + 0 &= 0 + mgL\,(1-\cos\theta_{\rm max}) \\
mgL\,(1-\cos\theta_{\rm max}) &= \frac{1}{2}mv_{\rm t}^2 \\
1-\cos\theta_{\rm max} &= \frac{v_{\rm t}^2}{2gL} \\
\cos\theta_{\rm max} &= 1-\frac{v_{\rm t}^2}{2gL} \\
\theta_{\rm max} &= \cos^{-1}\left(1-\frac{v_{\rm t}^2}{2gL}\right) \\
&= \cos^{-1}\left(1-\frac{(3.00\ {\rm m/s})^2}{2(9.80\ {\rm m/s^2})(0.800\ {\rm m})}\right) \\
&= \cos^{-1}(0.4260) = \boxed{64.8^\circ}\ .
\end{aligned}$$

Solution (c):



The diagram above shows the forces acting on the bob at the highest point with respect to an arbitrary coordinate system chosen such that the tension and total centripetal force points in the positive y direction. From this, we now construct a free-body diagram:



Since the tangential velocity at this highest point is zero [as mentioned in part (b)], the centripetal force must be zero:

$$F_c = ma_c = m\,\frac{v_t^2}{r} = 0\ .$$

Using this equilibrium equation in conjunction with the free-body diagram above, we get

$$\begin{aligned} F_c = \sum F_y &= T - F_{g\text{-}y} \\ 0 &= T - mg\cos\theta_{max} \\ T &= mg\cos\theta_{max} \\ &= (0.400\ \mathrm{kg})(9.80\ \mathrm{m/s^2})\ \cos(64.8^\circ) \\ &= \boxed{1.67\ \mathrm{N}\ .} \end{aligned}$$