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Automatica 41 (2005) 1–10
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Partial asymptotic stabilization of nonlinear distributed parameter systems☆

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Received 15 August 2003; received in revised form 27 July 2004; accepted 6 August 2004
Available online 6 October 2004

Abstract

The paper is devoted to stability and stabilization of a class of evolution equations arising from mathematical modeling of hybrid mechanical systems with flexible parts. A sufficient condition is obtained for partial strong asymptotic stability of nonlinear, infinite-dimensional dynamic systems in Banach spaces. This result is applied to deriving a control law that stabilizes a part of the variables describing a rotating rigid body endowed with a number of elastic beams. Results of numerical simulations are presented.

Keywords: Stabilization; Nonlinear control; Partial stability; Lyapunov function; Infinite-dimensional systems

1. Introduction

In recent years, the problems of modeling and control for mechanical systems consisting of coupled absolutely rigid and elastic parts have become an important research area. In particular, stability and stabilizability issues have been intensively studied for a rigid body endowed with the Euler–Bernoulli elastic beam in (Balas, 1978; Bloch & Titi, 1991; Coron & d'Andrea Novel, 1998; Krishnaprasad & Marsden, 1987; Laousy, Xu, & Sallet, 1996; Li, Chen, & Huang, 2000; Luo, Guo, & Morgul, 1999; Nabiullin, 1990; Xu & Baillieul, 1993).

The above class of hybrid systems is widely used in the engineering practice as controlled flexible manipulators, spacecrafts endowed with elastic antennae, solar panels, tethers, etc. Furthermore, motion of such systems is generally described by a set of coupled nonlinear ordinary and partial differential equations, that gives rise to a series of mathematical control theory problems concerning the evolution equations in infinite-dimensional spaces (Barbu, 1992; Curtain & Zwart, 1995; Fattorini, 1999).

It is a well-known fact (Rumyantsev & Oziraner, 1987; Vorotnikov, 1998) that many important classes of finite-dimensional mechanical systems cannot be stabilized in the sense of Lyapunov, while the concept of partial stability naturally appears. The goal of this paper is to characterize the partial stability and stabilizability of nonlinear infinite-dimensional differential equations describing the evolution of hybrid systems.

The paper is organized as follows. In Section 2, we consider the abstract Cauchy problem in a Banach space and prove the basic result on partial asymptotic stability. Our proof exploits an infinite-dimensional version of the Barbashin–Krasovskii and LaSalle invariance principle (Ball, 1978; Henry, 1981; LaSalle, 1976). Section 3 is devoted to mathematical modeling of the hybrid system consisting of a rigid body and several elastic beams. Within the framework of Lagrangian formalism, we obtain equations of motion for such an infinite-dimensional system. In Section 4, we construct a stabilizing feedback law for

☆A brief version of this paper was presented at the 15th IFAC World Congress in Barcelona, 2002. This paper was recommended for publication in revised form by Associate Editor George Weiss under the direction of Editor Roberto Tempo. The final text was written during the author's visit to the Technical University of Ilmenau, Institute for Automation and Systems Engineering, P.O.B. 100565, D-98684, Ilmenau, Germany.

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0005-1098/$ - see front matter © 2004 Elsevier Ltd. All rights reserved.
doi:10.1016/j.automatica.2004.08.009

the model considered. Some simulation results are presented in Section 5.

2. Partial strong asymptotic stability of infinite dimensional systems

Let X be a Banach space equipped with the norm $\|\cdot\|$. Assume that $D \subset X$ is a closed set containing some ball $B_R = \{x \in X \,|\, \|x\| \leqslant R\}$ of radius $R > 0$. Let F be a (nonlinear) operator from $\mathscr{D}(F) \subset D$ into X, F is supposed to be closed and densely defined on D. Given $x_0 \in D$, consider the abstract Cauchy problem (cf. Fattorini, 1999, Chap. 5.2)

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F(x(t)), \quad t \geqslant 0, \quad x(0) = x_0 \in D. \tag{1}$$

We assume that the operator F generates a nonlinear continuous semigroup on D.

Definition 1 (*Lakshmikantham and Leela, 1981, Chap. 2.8*). A *nonlinear semigroup* of operators on D is a one parameter family of mappings $\{S(t) \,|\, t \in \mathbb{R}_+\}$ from D into D such that

(i) $S(0)x = x$, for each $x \in D$, $S(t+s)x = S(t)S(s)x$, for $t, s \geqslant 0$ and $x \in D$, the function $t \mapsto S(t)x$ is continuous in $t \geqslant 0$ for each $x \in D$.

A nonlinear semigroup $\{S(t)\}$ is called *jointly continuous* (or simply *continuous* (Ladyzhenskaya, 1991, Chap. 1)) if the mapping $(t, x) \mapsto S(t)x$ from $\mathbb{R}_+ \times D \to D$ is continuous.

As F is the infinitesimal generator of a continuous semigroup $\{S(t)\}$, the Cauchy problem (1) is well posed, and any mild solution of (1) is given by

$$x(t) = S(t)x_0, \quad t \in \mathbb{R}_+, \quad x_0 \in D.$$

Now we introduce the notion of partial stability for the infinite dimensional system (1). Consider a bounded linear operator

$$\Pi : X \to X. \tag{2}$$

Definition 2. The semigroup S is said to be *strongly asymptotically stable with respect to* Π if

(i) given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|\Pi S(t)x_0\| < \varepsilon$, for all $t \geqslant 0$, there exists a $\Delta > 0$ such that $\|x_0\| < \Delta$ implies

$$\lim_{t\to+\infty} \|\Pi S(t)x_0\| = 0. \tag{3}$$

In particular, if Π is the identity operator, the above definition is equivalent to that of strong asymptotic stability. If Π projects X onto some linear subspace $X_1 \subset X$, then Definition 2 states asymptotic stability with respect to a part of the state variables parametrizing X_1.

We shall use the direct Lyapunov method in order to find out an effective condition ensuring partial stability of the infinite-dimensional autonomous system (1). Let $V : X \to \mathbb{R}$ be a differentiable functional in the sense of Fréchet. The time derivative of V along trajectories of (1) at $x \in D$ is defined as

$$\dot V(x) = \lim_{t\to+0} \frac{V(S(t)x) - V(x)}{t}.$$

The above expression can also be written in terms of vector fields for $x \in \mathscr{D}(F)$:

$$\dot V(x) = [\nabla V(x), F(x)], \tag{4}$$

where $[\cdot,\cdot] : X^* \times X \to \mathbb{R}$ is the duality pairing of X and X^*, i.e. $[\nabla V(x), y]$ is the value of a linear functional $\nabla V \in X^*$ at $y \in X$. In particular, when X is a Hilbert space (or a finite dimensional Euclidean space), (4) takes the following form:

$$\dot V(x) = \langle \nabla V(x), Fx \rangle. \tag{5}$$

Here $\langle\cdot,\cdot\rangle$ is the scalar product in X.

To formulate a sufficient condition for partial stability, we introduce the standard class $\mathscr{K}$ consisting of all continuous strictly increasing functions $\alpha : [0, +\infty) \to [0, +\infty)$ such that $\alpha(0) = 0$.

Theorem 1. *Let F be the infinitesimal generator of a nonlinear continuous semigroup $\{S(t)\}$ on D, $F(0) = 0$, and let $\Pi : X \to X$ be a continuous linear operator. Assume that there exists a differentiable functional $V : X \to \mathbb{R}_+$ satisfying the following conditions*:

(1) *There exist $\alpha_1(\cdot), \alpha_2(\cdot) \in \mathscr{K}$ such that*

$$\alpha_1(\|\Pi x\|) \leqslant V(x) \leqslant \alpha_2(\|x\|) \quad \textit{for all } x \in X. \tag{6}$$

$$\dot V(x) \leqslant 0 \quad \textit{for all } x \in \mathscr{D}(F). \tag{7}$$

There exists $\Delta > 0$ such that

$$\bigcup_{t\geqslant 0} \{S(t)x_0\}$$

is precompact (*relatively compact*) *provided that $\|x_0\| < \Delta$. The set*

$$M = \overline{\{x \in \mathscr{D}(F) \,|\, \dot V(x) = 0\}} \backslash \mathrm{Ker}\, \Pi$$

does not contain any semitrajectory of (1) *defined for $t \in \mathbb{R}_+$. The kernel of Π is a forward and backward invariant for* (1), *i.e. $\Pi S(\tau)x_0 = 0$ for some $x_0 \in X$ and $\tau \geqslant 0$ implies $\Pi S(t)x_0 = 0$ for all $t \geqslant 0$.*

Then system (1) *is strongly asymptotically stable with respect to Π.*

The proof is divided into two parts. First, we prove stability with respect to Π by extending the Rumyantsev theorem (Rumyantsev & Oziraner, 1987, Theorem 5.1; Vorotnikov, 1998); for the case of a Banach space. Then we apply an infinite-dimensional modification of the invariance principle due to Barbashin–Krasovskii and LaSalle.

Let us show that inequality (7) implies $V(x(t))$ is nonincreasing on any mild solution of (1) with $x_0 \in D$, $t \in \mathbb{R}_+$. If $x_0 \in \mathscr{D}(F)$, then $x(t) = S(t)x_0$ is a classical solution, and $\dot{V}(x(t))$ given by formula (4) exists for all $t \geqslant 0$. As $V(x(t))$ is continuous and $\dot{V}(x(t)) \leqslant 0$ on $\mathbb{R}_+$, $V(x(t))$ is nonincreasing on $\mathbb{R}_+$. For arbitrary $x_0 \in D \backslash \mathscr{D}(F)$ and $T > 0$, the mild solution $S(t)x_0$ $(0 \leqslant t \leqslant T)$ can be approximated by classical ones in the $L^\infty([0, T]; X)$ norm due to continuity of $S(t)$. Therefore, as V is nonincreasing along any classical solution and V is continuous, $V(S(t)x_0)$ is nonincreasing on $\mathbb{R}_+$, so we have

$$V(S(t)x_0) \leqslant V(x_0), \quad \forall x_0 \in D, \ t \geqslant 0.$$

By combining the above inequality with (6), we get

$$\|\Pi S(t)x_0\| \leqslant \alpha_1^{-1}(\alpha_2(\|x_0\|)), \quad t \geqslant 0, \tag{8}$$

where the function $\alpha_1^{-1}(p)$ exists and increases at least for small enough $p > 0$, since $\alpha_1(\cdot)$ is strictly increasing. Therefore, the function

$$\gamma(\delta) = \alpha_1^{-1}(\alpha_2(\delta))$$

is continuous, nonnegative, and strictly increasing on some interval $[0, \delta^*)$, $0 < \delta^* \leqslant +\infty$. It means that for any $\varepsilon > 0$ there exists $\delta \in (0, \delta^*)$ such that $\gamma(\delta) \leqslant \varepsilon$. Hence, if $\|x_0\| < \delta$ then (8) implies

$$\|\Pi S(t)x_0\| \leqslant \gamma(\|x_0\|) < \gamma(\delta) \leqslant \varepsilon, \quad t \geqslant 0.$$

In order to finish the proof, it is sufficient to establish the limit existence for (3). Let $\|x_0\| < \Delta$. Condition 3 implies that the corresponding semitrajectory

$$\pi^+(x_0) = \{S(t)x_0\}_{t \geqslant 0}$$

has a nonempty and compact ω-limit set, say $\Omega(x_0)$. LaSalle's Theorem (LaSalle, 1976) implies

$$\Omega(x_0) \subseteq \Theta,$$

where Θ is the largest invariant subset of

$$Z = \overline{\{x \in \mathscr{D}(F) \mid \dot{V}(x) = 0\}}.$$

(The closure is taken in Z because (4) defines $\dot{V}$ on $\mathscr{D}(F)$ only, which is dense in D.) Since the operator Π is continuous, it suffices to show that

$$\Theta \subseteq \operatorname{Ker} \Pi. \tag{9}$$

To prove (9), let us suppose the contrary. Assume that there exists some $\tilde{x} \in \Theta$ such that $\Pi\tilde{x} \neq 0$. As Θ is an invariant subset of Z and $\tilde{x} \in \Theta$, then

$$S(t)\tilde{x} \in Z \quad \text{for all } t \geqslant 0 \ (\Pi\tilde{x} \neq 0). \tag{10}$$

If $\Pi S(t)\tilde{x} \neq 0$ for all $t \in [0, +\infty)$ then (10) contradicts condition (4). Otherwise, there exists $T > 0$ such that $\Pi S(T)\tilde{x} = 0$, $\Pi S(0)\tilde{x} \neq 0$. But the above contradicts condition (5). Therefore,

$$\lim_{t \to +\infty} \|\Pi S(t)x_0\| = 0,$$

provided that $\|x_0\| < \Delta$. $\square$

Remark. For the finite-dimensional case, when $x \in X = \mathbb{R}^n$ and

$$\Pi : (x_1, x_2, \ldots, x_n) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0), \ (m \leqslant n),$$

the above theorem is analogous to the result by C. Risito and V.V. Rumyantsev (Rumyantsev & Oziraner, 1987, p. 99). For this finite-dimensional framework, a C_0-controller design scheme has been proposed by Zuyev (2000).

3. Modeling of a rigid body with elastic attachments

In this chapter, we consider a mechanical system on the plane consisting of a rigid body and k elastic beams (Fig. 1). This system is a rough model of a controlled satellite with flexible antennae performing planar maneuvers; see Nabiullin (1990) for more complicated models.

The rigid body can rotate around a fixed point O under the action of a control torque ζ. We suppose that each beam is attached at distance d from the point O, and that l is the length of the beams. Let $w_i(x, t)$ be the deflection of the ith beam from the axis $O_i\xi_i$ at the location $x \in [0, l]$ and time $t \geqslant 0$. Thus, the evolution of the system is defined by the following functions:

$$\theta(t), \ w_i(x, t), \qquad x \in [0, l], \ t \geqslant 0, \ i = \overline{1, k},$$

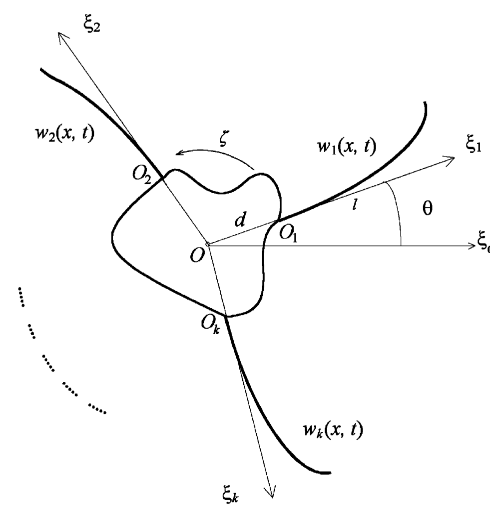

Fig. 1. The body-beams system.

where θ is the angle between the axes $O\xi_0$ and $O\xi_1$. As each beam is fixed at one end, while the other end is free, we have the following boundary conditions (see, e.g., Baillieul & Levi, 1991):

$$w_i|_{x=0} = \left.\frac{\partial w_i}{\partial x}\right|_{x=0} = \left.\frac{\partial^2 w_i}{\partial x^2}\right|_{x=l} = \left.\frac{\partial^3 w_i}{\partial x^3}\right|_{x=l} = 0. \quad (11)$$

We follow here the Euler–Bernoulli beam theory and assume $w_i \in H$ for each $i = \overline{1,k}$,

$$H = \{w(x,t) \mid w(\cdot,t) \in L^2(0,l),\, w_t(\cdot,t) \in L^2(0,l),\, w_{xx}(\cdot,t) \in L^2(0,l),\, w(0,t) = w_x(0,t) = 0,\, \forall t \geqslant 0\}. \quad (12)$$

To derive the equations of motion of this mechanical system, we expand the deflections $w_i(\cdot,t)$ with respect to a basis $\{u_n(\cdot)\}$ of $L^2(0,l)$ as

$$w_i(x,t) = \sum_{n=1}^{\infty} u_n(x) q_{in}(t), \quad (13)$$

where $q_{in}(t)$ is a generalized elastic coordinate corresponding to the nth mode. We choose $\{u_n(\cdot)\}$ by solving the following Sturm–Liouville problem:

$$L[u] \equiv \frac{d^4}{dx^4} u(x) = \lambda u(x), \quad x \in (0,l),$$

$$u(0) = u'(0) = u''(l) = u'''(l) = 0. \quad (14)$$

It is well-known (see e.g., Krylov, 1911; Luo et al., 1999) that the above problem has the sequence of eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

with

$$\lambda_n = O(n^4) \quad \text{as } n \to \infty, \quad (15)$$

and the corresponding eigenfunctions $\{u_n(\cdot)\}_{n=1}^{\infty}$ forming an orthogonal basis in $L^2(0,l)$. Therefore, each $w_i \in H$ satisfying (11) admits a unique representation (13). Let $\{(\beta_n^4, \phi_n)\}_{n=1}^{\infty}$ be the eigenvalues and eigenfunctions of (14) for $l = 1$. Then $\beta_n > 0$ are the solutions of

$$1 + \cos(\beta_n)\cosh(\beta_n) = 0,$$

and ϕ_n can be expressed as follows (Luo et al., 1999, p. 176):

$$\phi_n(x) = -\frac{1+\gamma_n}{2} e^{\beta_n x} - \frac{1-\gamma_n}{2} e^{-\beta_n x} + \gamma_n \sin(\beta_n x) + \cos(\beta_n x),$$

$$\gamma_n = -\frac{e^{\beta_n} - \sin\beta_n + \cos\beta_n}{e^{\beta_n} + \sin\beta_n + \cos\beta_n} < 0. \quad (16)$$

For arbitrary $l > 0$, we have $\lambda_n = (\beta_n/l)^4$ and $u_n(x) = k_n \phi_n(x/l)$, where the constants k_n are defined to normalize u_n in $L^2(0,l)$:

$$\|u_n\|^2_{L^2(0,l)} = \int_0^l u_n^2(x)\, dx = 1.$$

This implies

$$k_n = \pm l^{-1/2} (\|\phi_n\|_{L^2(0,1)})^{-1}. \quad (17)$$

The Lagrangian of the system considered is

$$2L = J\dot\theta^2 + \rho \sum_{n=1}^{\infty}\sum_{i=1}^{k} \left(\dot q_{in}^2 + \frac{2J_n}{\rho}\dot\theta \dot q_{in} + \dot\theta^2 q_{in}^2 - c^2 \lambda_n q_{in}^2 \right),$$

where $J_0 > 0$ is the inertia moment of the rigid body, ρ – the density of the beams, ρc^2 – the flexural rigidity per unit length, $J = J_0 + k\rho l(\frac{1}{3}l^2 + ld + d^2)$, and

$$J_n = \rho \int_0^l (x+d) u_n(x)\, dx = \rho l^2 k_n \int_0^1 s\phi_n(s)\, ds + \rho l d k_n \int_0^1 \phi_n(s)\, ds. \quad (18)$$

Here ρ and c are assumed to be positive constants. It follows from Lemma 4.6 of (Luo et al., 1999, p. 176) that

$$\int_0^1 s\phi_n(s)\, ds = -2\beta_n^{-2} < 0, \qquad \int_0^1 \phi_n(s)\, ds = 2\gamma_n \beta_n^{-1} < 0.$$

Therefore,

$$J_n = 2\rho l k_n \beta_n^{-1}(d\gamma_n - l\beta_n^{-1}) = O(n^{-1}) \neq 0. \quad (19)$$

Now we can write the Lagrange equations of motion (see, e.g., Goldstein, 1980; Nabiullin, 1990 for more details):

$$J\ddot\theta + \sum_{n=1}^{\infty}\sum_{i=1}^{k} (\rho\ddot\theta q_{in}^2 + 2\rho\dot\theta q_{in}\dot q_{in} + J_n \ddot q_{in}) = \zeta,$$

$$J_n\ddot\theta + \rho\ddot q_{in} + \rho(\lambda_n c^2 - \dot\theta^2) q_{in} = 0 \quad (i = \overline{1,n},\ n = 1,2,\ldots). \quad (20)$$

To simplify (20), we replace the control torque ζ applied to the rigid body with a new control v by means of the following feedback transformation:

$$v = \frac{\zeta + \sum_{i=1}^{k}\sum_{n=1}^{\infty}[J_n(\lambda_n c^2 - \dot\theta^2) - 2\rho\dot\theta \dot q_{in}] q_{in}}{J - \frac{k}{\rho}\sum_{n=1}^{\infty} J_n^2 + \rho\sum_{i=1}^{k}\sum_{n=1}^{\infty} q_{in}^2}. \quad (21)$$

The denominator of the above transformation is bounded from below by

$$J - \frac{k}{\rho}\sum_{n=1}^{\infty} J_n^2 = J_0 > 0. \quad (22)$$

Formula (22) follows from (18) and Parseval's identity for $\{u_n(\cdot)\}$:

$$\frac{1}{\rho^2}\sum_{n=1}^{\infty} J_n^2 = \int_0^l (x+l)^2\, dx = l\left(\frac{1}{3}l^2 + ld + d^2\right) = \frac{J - J_0}{k\rho}.$$

As the deflections $w_i(x,t)$ are considered in the space H defined by (12), it imposes the following constraint on q_{in} and $\dot{q}_{in}$:

$$\sum_{i=1}^{k}\sum_{n=1}^{\infty}\lambda_n(q_{in})^2<\infty,\quad \sum_{i=1}^{k}\sum_{n=1}^{\infty}\dot{q_{in}}^2<\infty.$$

To satisfy this requirement, we introduce new variables

$$\xi_{in}=c\sqrt{\lambda_n}q_{in},\quad \eta_{in}=\dot{q}_{in} \tag{23}$$

and assume

$$\sum_{i=1}^{k}\sum_{n=1}^{\infty}(\xi_{in}^2+\eta_{in}^2)<\infty. \tag{24}$$

By applying the feedback transformation (21), (23) in (20), we get the following control system:

$$\dot{x}=Ax+R(x)x+bv,\quad x\in\ell^2,\ v\in\mathbb{R}, \tag{25}$$

where

$$x=(\theta,\omega,\ \xi_{11},\eta_{11},\dots,\xi_{k1},\eta_{k1},\ \xi_{12},\eta_{12},\dots)^{\mathrm{T}},$$

and ℓ^2 is the Hilbert space of all infinite vectors x with real coordinates satisfying (24). Let us recall that for each $x\in\ell^2$ and

$$\bar{x}=(\bar\theta,\bar\omega,\ \bar\xi_{11},\bar\eta_{11},\dots,\bar\xi_{k1},\bar\eta_{k1},\ \bar\xi_{12},\bar\eta_{12},\dots)^{\mathrm{T}}\in\ell^2,$$

the scalar product in ℓ^2 is defined as follows

$$\langle x,\bar{x}\rangle_{\ell^2}=\theta\bar\theta+\omega\bar\omega+\sum_{i=1}^{k}\sum_{n=1}^{\infty}(\xi_{in}\bar\xi_{in}+\eta_{in}\bar\eta_{in}).$$

Operators A and R in (25) are given by their matrices

$$A=\mathrm{diag}(A_0,A_1,\dots,A_1,A_2,\dots),$$

$$R(x)=\mathrm{diag}(O_{2\times2},R_1(x),\dots,R_1(x),R_2(x),\dots),$$

$$b=\left(0,1,\ 0,-\frac{J_1}{\rho},\dots,0,-\frac{J_1}{\rho},\ 0,0,-\frac{J_2}{\rho},\dots\right)^{\mathrm{T}}\in\ell^2.$$

Here

$$A_0=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad A_n=\begin{pmatrix}0&c\sqrt{\lambda_n}\\-c\sqrt{\lambda_n}&0\end{pmatrix},$$

$$R_n(x)=\begin{pmatrix}0&0\\\frac{\omega^2}{c\sqrt{\lambda_n}}&0\end{pmatrix},\quad n\in\mathbb{N}.$$

It is easy to see that

$$\mathscr{D}(A)=\left\{x\in\ell^2\ \middle|\ \sum_{i=1}^{k}\sum_{n=1}^{\infty}\lambda_n(\xi_{in}^2+\eta_{in}^2)<\infty\right\},$$

so the operator A is densely defined in ℓ^2. The feedback transformation (21) is well defined for all $x\in\mathscr{D}(A)$ as its numerator is finite. Indeed, the Cauchy–Schwartz inequality together with estimates (15) and (19) yields

$$\left|\sum_{i=1}^{k}\sum_{n=1}^{\infty}J_n(\lambda_nc^2-\dot\theta^2)q_{in}\right|$$
$$\leqslant\left(k\sum_{n=1}^{\infty}J_n^2\right)^{1/2}\left(\sum_{i=1}^{k}\sum_{n=1}^{\infty}\left(c^2\lambda_n+\frac{\omega^4}{c^2\lambda_n}\right)\xi_{in}^2\right)^{1/2}$$
$$<\infty,$$
$$\left|\sum_{i=1}^{k}\sum_{n=1}^{\infty}q_{in}\dot{q}_{in}\right|\leqslant\left(\sum_{i=1}^{k}\sum_{n=1}^{\infty}\frac{\xi_{in}^2}{c^2\lambda_n}\right)^{1/2}\left(\sum_{i=1}^{k}\sum_{n=1}^{\infty}\eta_{in}^2\right)^{1/2}$$
$$<\infty,$$

provided that $x\in\mathscr{D}(A)$.

4. Attitude stabilization

In this section, we apply Theorem 1 to derive a feedback control that ensures partial strong asymptotic stability of (25).

Let us show first that the linear approximation of (25) is not stabilizable for $k\geqslant2$. Indeed, consider the diffeomorphism $\phi:\ell^2\to\ell^2$ such that

$$\phi(x)=(\theta,\omega,\dots,Q_{in},P_{in},\dots)^{\mathrm{T}},$$

$$Q_{kn}=\frac{1}{k}\sum_{j=1}^{k}\xi_{jn},\quad P_{kn}=\frac{1}{k}\sum_{j=1}^{k}\eta_{jn},$$

$$Q_{in}=\xi_{in}-Q_{kn},\quad P_{in}=\eta_{in}-P_{kn},\quad i=\overline{1,k-1}.$$

By applying ϕ into the linear part of (25), we get the following dynamics on Q_{in}, P_{in}:

$$\dot{Q}_{in}=c\sqrt{\lambda}P_{in},$$
$$\dot{P}_{in}=-c\sqrt{\lambda}Q_{in}\quad(i=\overline{1,k-1},\ n\geqslant1). \tag{26}$$

We have, for any solution of (26),

$$\sum_{i=1}^{k-1}\sum_{n=1}^{\infty}(P_{in}(t)^2+Q_{in}(t)^2)=\mathrm{const}. \tag{27}$$

It means that subsystem (26) and, therefore, the linear approximation of (25) cannot be made asymptotically stable if $k\geqslant2$. To overcome this obstruction, we define the bounded linear operator $\Pi:\ell^2\to\ell^2$ as follows:

$$\Pi:x$$
$$\mapsto\left(\theta,\omega,\ \sum_{i=1}^{k}\xi_{i1},\sum_{i=1}^{k}\eta_{i1},\ \dots,\ \sum_{i=1}^{k}\xi_{in},\sum_{i=1}^{k}\eta_{in},\ \dots\right)^{\mathrm{T}}. \tag{28}$$

If $k=1$ then Π is the identity operator on ℓ^2, otherwise $\|\Pi\phi^{-1}(\theta,\omega,\dots,Q_{in},P_{in},\dots,)^{\mathrm{T}}\|_{\ell^2}$ does not depend on

Q_{in}, P_{in} with $i \leqslant k-1$, therefore, the value of (27) does not affect $\|\Pi x(t)\|_{\ell^2}$ in the linear case.

As we see, the operator Π projects the state space onto the linear subspace parametrized by the angular position and angular velocity of the rigid body together with all coordinates Q_{kn} and P_{kn}. Hence, Πx describes the state of an "averaged" body-beam system having just one beam, and the flexible coordinates of that system are obtained by averaging the coordinates with the same mode number n for all k beams of the original system. Stabilization with respect to Π leads, from the physical point of view, to oscillation damping for the rigid body and simultaneous damping of the averaged beams positions and velocities. This does not damp a priori the beams offsets from the averaged positions, i.e. the coordinates Q_{in} and P_{in} with $i \leqslant k-1$.

Theorem 2. *There exists a feedback law ensuring strong asymptotic stability of the nonlinear system* (25) *with respect to operator* (28). *In addition, the feedback proposed preserves strong* (*non-asymptotic*) *stability of the equilibrium* $x=0$ *in* ℓ^2, *and the Cauchy problem for the closed-loop system on* $t \geqslant 0$ *is well posed* (*in the sense of mild solutions*) *in a neighborhood of* $x=0$.

The rest of this section is devoted to the proof of Theorem 2.

4.1. Lyapunov functional design

Since the original system (20) is Lagrangian, we shall use the following energy-based Lyapunov functional:

$$2V(x) = c_0\theta^2 + J\omega^2 + \sum_{i=1}^{k}\sum_{n=1}^{\infty} \times (\rho\xi_{in}^2 + \rho\eta_{in}^2 + 2J_n\omega\eta_{in}). \tag{29}$$

By applying the Cauchy–Schwartz inequality, we get

$$\left|\sum_{i=1}^{k}\sum_{n=1}^{\infty} J_n\eta_{in}\right| \leqslant \sqrt{k}\left(\sum_{n=1}^{\infty} J_n^2\right)^{1/2}\left(\sum_{i=1}^{k}\sum_{n=1}^{\infty}\eta_{in}^2\right)^{1/2}.$$

The above inequality together with (29) implies

$$G\left(-|\omega|, \left(\sum_{i,n}\eta_{in}^2\right)^{1/2}\right) \leqslant 2V(x) - c_0\theta^2 - \rho\sum_{i,n}\xi_{in}^2 \leqslant G\left(|\omega|, \left(\sum_{i,n}\eta_{in}^2\right)^{1/2}\right), \tag{30}$$

where the quadratic form G is defined as follows:

$$G(a,b) = Ja^2 + 2\sqrt{k}\left(\sum_{n=1}^{\infty} J_n^2\right)^{1/2} ab + \rho b^2.$$

By computing the eigenvalues for the matrix of G, we get

$$c_1(a^2+b^2) \leqslant G(a,b) \leqslant c_2(a^2+b^2), \tag{31}$$

where

$$c_1 = \tfrac{1}{2}(\rho + J - \sqrt{(\rho+J)^2 - 4\rho J_0}) > 0,$$

$$c_2 = \tfrac{1}{2}\left(\rho + J + \sqrt{(\rho+J)^2 - 4\rho J_0}\right) > 0. \tag{32}$$

Then (29)–(31) yield, for all $x \in \ell^2$,

$$\min\{c_0, c_1, \rho\}\|x\|_{\ell^2}^2 \leqslant 2V(x) \leqslant \max\{c_0, c_2, \rho\}\|x\|_{\ell^2}^2. \tag{33}$$

Therefore, V satisfies the first condition of Theorem 1 for any bounded linear operator $\Pi : \ell^2 \to \ell^2$.

We use expression (5) to calculate $\dot V$:

$$\dot V = \left(c_0\theta + \frac{1}{c}\sum_{i=1}^{k}\sum_{n=1}^{\infty}\frac{\xi_{in}\left(J_n(\omega^2 - c^2\lambda_n) + \rho\omega\eta_{in}\right)}{\sqrt{\lambda_n}}\right)\omega + \left(J - \frac{k}{\rho}\sum_{n=1}^{\infty}J_n^2\right)\omega v. \tag{34}$$

As $\dot V$ divides by ω, a natural candidate for a stabilizing feedback $v(x)$ is obtained by assuming $\dot V = -h\omega^2 + o(\|x\|_{\ell^2}^2)$, where h is a positive constant. This can be done with a linear feedback as follows:

$$v(x) = -\frac{1}{J_0}\left(c_0\theta + h\omega - c\sum_{i=1}^{k}\sum_{n=1}^{\infty}\sqrt{\lambda_n}J_n\xi_{in}\right). \tag{35}$$

The time derivative of V along the solution of (25), (35) is

$$\dot V = -h\omega^2\left(1 - \frac{1}{ch}\sum_{i=1}^{k}\sum_{n=1}^{\infty}\frac{\xi_{in}(J_n\omega + \rho\eta_{in})}{\sqrt{\lambda_n}}\right). \tag{36}$$

Then the second condition of Theorem 1 holds provided that

$$\sum_{i=1}^{k}\sum_{n=1}^{\infty}\frac{\xi_{in}(J_n\omega + \rho\eta_{in})}{\sqrt{\lambda_n}} < ch. \tag{37}$$

By applying the Cauchy–Schwartz inequality in (37) together with (33), we conclude that $\dot V \leqslant 0$ holds in

$$D = \left\{x \in \ell^2 \mid V(x) \leqslant \mu\left(\left(k\sum_{n=1}^{\infty}\frac{J_n^2}{\lambda_n}\right)^{1/2} + \frac{\rho}{\sqrt{\lambda_1}}\right)^{-1}\right\} \tag{38}$$

for any positive constant $\mu < \frac{1}{2}ch\min\{c_0, c_1, \rho\}$. The above defined D is positively invariant for (25), (35) as it is described by level sets of the Lyapunov function V. Moreover, the equilibrium $x_0 = 0$ of the closed-loop system is (non-asymptotically) stable because of estimate (33) and the inequality $\dot V \leqslant 0$ in D.

In order to justify that the closed-loop system generates a continuous semigroup on D, let us first consider the linear approximation of (25), (35):

$$\dot{x}(t) = -\tilde{A}x(t), \quad x(0) = x_0 \in \ell^2, \tag{39}$$

where $-\tilde{A}x = Ax + bv(x)$, $\overline{\mathscr{D}(\tilde{A})} = \overline{\mathscr{D}(A)} = \ell^2$. It is easy to check that the operator $\tilde{A}$ is accretive in ℓ^2 equipped with the following scalar product:

$$\langle x, \bar{x}\rangle_Q = c_0\theta\bar{\theta} + J\omega\bar{\omega} + \sum_{i=1}^{k}\sum_{n=1}^{\infty} J_n\left(\omega\bar{\eta}_{in} + \bar{\omega}\eta_{in}\right) + \rho\sum_{i=1}^{k}\sum_{n=1}^{\infty}\left(\xi_{in}\bar{\xi}_{in} + \eta_{in}\bar{\eta}_{in}\right).$$

The norm $\|x\|_Q = (\langle x, x\rangle_Q)^{1/2}$ is equivalent to the standard norm $\|x\|_{\ell^2} = (\langle x, x\rangle_{\ell^2})^{1/2}$ because of estimate (33). Corollary 4.4 of Pazy (1983, p. 15) implies that $-\tilde{A}$ is the infinitesimal generator of the C_0 semigroup of contractions $\{e^{-t\tilde{A}} \mid t \geqslant 0\}$ on $(\ell^2, \langle\cdot,\cdot\rangle_Q)$, and therefore, on $(\ell^2, \langle\cdot,\cdot\rangle_{\ell^2})$.

As the right-hand side of (25), (35) is obtained by a locally Lipschitz perturbation of the infinitesimal generator $-\tilde{A}$ then, for each $x_0 \in D$, there is the mild solution $x(t)$ of (25), (35) defined on the maximal interval $[0, t_{\max})$ (Pazy, 1983, Theorem 1.4, p. 185). Moreover, $t_{\max} = +\infty$ because otherwise we would have

$$\lim_{t\to t_{\max}} \|x(t)\|_{\ell^2} = \infty,$$

which contradicts the solutions boundedness in D provided by the inequality $\dot{V} \leqslant 0$. Thus, the closed-loop system generates a continuous semigroup on D (continuity follows from (Pazy, 1983, Chap. 6)). We have proved that for each $x(0) \in D$, the abstract Cauchy problem for the closed-loop system, a unique mild solution $x(t)$ was defined on $t \geqslant 0$.

From (36) and (38) we get

$$\begin{aligned} V(x_0) &\geqslant V(x_0) - \lim_{t\to+\infty} V(x(t)) = -\int_0^{+\infty} \dot{V}(x(t))\,dt \\ &\geqslant h\int_0^{+\infty} \omega^2(t)\,dt \inf_{x\in D} \\ &\quad\times\left(1 - \frac{1}{ch}\sum_{i=1}^{k}\sum_{n=1}^{\infty}\frac{\xi_{in}(J_n\omega + \rho\eta_{in})}{\sqrt{\lambda_n}}\right) \\ &\geqslant C^*\int_0^{+\infty}\omega^2(t)\,dt \end{aligned}$$

with some positive constant C^* provided that $x_0 \in D$. Therefore,

$$\int_0^{+\infty}\omega^2(t)\,dt < \infty \tag{40}$$

for any closed-loop solution starting in D.

4.2. Precompactness analysis

We apply Dafermos and Slemrod (1973, Theorem 3) to show the trajectories' precompactness for (39). It is necessary to prove that $(\lambda\tilde{A} + I)^{-1}$ is compact in ℓ^2. For this purpose we first compute the solutions of linear equation $Ix - \lambda Ax - \lambda bv = \bar{x}$ with respect to x assuming λ, v, and $\bar{x}$ to be parameters. Straightforward calculations yield

$$\omega = \bar{\omega} + \lambda v, \quad \theta = \bar{\theta} + \lambda\bar{\omega} + \lambda^2 v, \tag{41}$$

$$\begin{pmatrix}\xi_{in}\\ \eta_{in}\end{pmatrix} = \frac{1}{1+\lambda^2c^2\lambda_n}\begin{pmatrix}1 & \lambda c\sqrt{\lambda_n}\\ -\lambda c\sqrt{\lambda_n} & 1\end{pmatrix} \times\begin{pmatrix}\bar{\xi}_{in}\\ \bar{\eta}_{in} - \frac{\lambda J_n}{\rho}v\end{pmatrix}. \tag{42}$$

Then by substituting the above expressions into (35), we get the following relation on v:

$$\begin{aligned} v = &-\left(J_0 + \lambda h + \lambda^2 c_0 + \frac{\lambda^2c^2}{\rho}\sum_{i=1}^{k}\sum_{n=1}^{\infty}\frac{\lambda_n J_n^2}{1+\lambda^2c^2\lambda_n}\right)^{-1} \\ &\times\left(c_0\bar{\theta} + (\lambda c_0 + h)\bar{\omega} - c\sum_{i=1}^{k}\sum_{n=1}^{\infty}\right. \\ &\left.\times\frac{J_n\sqrt{\lambda_n}(\bar{\xi}_{in} + \lambda c\sqrt{\lambda_n}\bar{\eta}_{in})}{1+\lambda^2c^2\lambda_n}\right). \end{aligned} \tag{43}$$

Formula (43) defines the linear functional $v(\bar{x})$ on ℓ^2. For arbitrary $\lambda > 0$, $v(\bar{x})$ is bounded as its coefficients belong to ℓ^2:

$$0 < J_0 + \lambda h + \lambda^2 c_0 + \frac{\lambda^2c^2}{\rho}\sum_{i=1}^{k}\sum_{n=1}^{\infty}\frac{\lambda_n J_n^2}{1+\lambda^2c^2\lambda_n} = \text{const} < \infty,$$

$$\sum_{i=1}^{k}\sum_{n=1}^{\infty}\frac{J_n^2\lambda_n}{(1+\lambda^2c^2\lambda_n)^2} < \infty, \quad \sum_{i=1}^{k}\sum_{n=1}^{\infty}\frac{J_n^2\lambda_n^2}{(1+\lambda^2c^2\lambda_n)^2} < \infty.$$

The above series converge due to estimates (15) and (19). Therefore, for each $\lambda > 0$, there is a positive $M_1(\lambda)$ such that $|v(\bar{x})| \leqslant M_1(\lambda)\|\bar{x}\|_{\ell^2}$ in (43) for all $\bar{x} \in \ell^2$.

Formulae (41)–(43) define $x = (\lambda A + I)^{-1}\bar{x}$ for all $\bar{x} \in \ell^2$ provided that $\lambda > 0$. It is easy to see that

$$\begin{aligned} \|(\lambda A + I)^{-1}\bar{x}\|_{\ell^2}^2 &\leqslant (\bar{\omega} + \lambda v(\bar{x}))^2 + (\bar{\theta} + \lambda\bar{\omega} + \lambda^2 v(\bar{x}))^2 \\ &+ 2k\left(\sum_{n=1}^{\infty}\frac{1}{1+\lambda^2c^2\lambda_n}\right)\sum_{i,n}(\bar{\xi}_{in}^2 \\ &+ (\bar{\eta}_{in} - \lambda J_n v(\bar{x})/\rho)^2) \leqslant M_2(\lambda)\|\bar{x}\|_{\ell^2}^2 \end{aligned}$$

for some $M_2(\lambda)$ as

$$\sum_{i=1}^{k}\sum_{n=1}^{\infty}\frac{1}{1+\lambda^2c^2\lambda_n} < \infty \tag{44}$$

and $v(\bar{x})$ is bounded.

Therefore, the resolvent $(\lambda A+I)^{-1}: \ell^2 \to \ell^2$ is bounded for each $\lambda>0$. To prove its compactness let us define the operator $\Pi_N: \ell^2 \to \ell^2$ that projects each x onto the subspace spanned by the elements of ℓ^2 having $\theta=\omega=\xi_{in}=\eta_{in}=0$ for all $n<N$. In the coordinate form Π_N can be expressed as

$$\Pi_N x = (0,0,\dots,0,\ \xi_{1N},\eta_{1N},\dots,\xi_{kN},\eta_{kN},\ \xi_{1,N+1},\dots)^{\mathrm{T}}.$$

Consider the following sequence of bounded linear operators on ℓ^2:

$$U_N = (I-\Pi_N)(\lambda A+I)^{-1}.$$

Each U_N is compact because the dimension of its image is finite. By Theorem 3 of (Kantorovich & Akilov, 1982, p. 246), the operator $(\lambda A+I)^{-1}$ is compact if

$$\lim_{N\to\infty}\|(\lambda A+I)^{-1}-U_N\| = \lim_{N\to\infty}\|\Pi_N(\lambda A+I)^{-1}\|=0. \tag{45}$$

We have

$$\begin{aligned}&\frac{1}{2k}\left(\sum_{n=N}^{\infty}\frac{1}{1+\lambda^2c^2\lambda_n}\right)^{-1}\|\Pi_N(\lambda A+I)^{-1}\bar{x}\|^2_{\ell^2}\\ &\leqslant \sum_{i=1}^{k}\sum_{n=1}^{\infty}\left(\bar{\xi}_{in}^{\,2}+(\bar{\eta}_{in}-\lambda J_n v(\bar{x})/\rho)^2\right)\\ &\leqslant\left(\left(1+\frac{2\lambda M_1(\lambda)}{\rho}\right)\left(k\sum_{n=1}^{\infty}J_n^2\right)^{1/2}\right.\\ &\quad\left.+\frac{\lambda^2M_1(\lambda)^2k}{\rho^2}\sum_{n=1}^{\infty}J_n^2\right)\|\bar{x}\|^2_{\ell^2}.\end{aligned} \tag{46}$$

Convergency of (44) implies

$$\sum_{n=N}^{\infty}\frac{1}{1+\lambda^2c^2\lambda_n}\to 0 \quad \text{as } N\to\infty.$$

Now (45) follows from (46). Hence, $(\lambda A+I)^{-1}$ is compact for each $\lambda>0$, and the trajectories of the semigroup $\{e^{-tA}\mid t\geqslant 0\}$ are compact due to Dafermos and Slemrod (1973, Theorem 3).

To establish precompactness of the trajectories for the nonlinear closed-loop system (25), (35) we write it in the following form:

$$\dot{x}(t) = -\tilde{A}x(t)+\omega^2(t)Bx(t),\quad x(0)=x_0\in\ell^2, \tag{47}$$

where

$$Bx(t)=\left(0,0,\dots,0,\frac{\xi_{in}(t)}{c\sqrt{\lambda_n}},\dots\right)^{\mathrm{T}}.$$

To characterize precompact subsets of ℓ^2 we need one preliminary.

Lemma 1. *A bounded set $C\subset\ell^2$ is precompact if and only if, for every $\varepsilon>0$, there is a number $N=N(\varepsilon)$ such that*

$$\|\Pi_N x\|_{\ell^2}<\varepsilon,\quad \forall x\in C. \tag{48}$$

The assertion of Lemma follows from the Hausdorff compactness criterion (Kantorovich & Akilov, 1982, Theorem 3, p. 26). □

The solutions of (47) satisfy the integral equation

$$x(t)=e^{-t\tilde{A}}x_0+\int_0^t e^{(\tau-t)\tilde{A}}\omega^2(\tau)Bx(\tau)\,d\tau.$$

It follows from Lemma 1 and precompactness of $\{e^{-t\tilde{A}}x_0\mid t\geqslant 0\}$ that, given $x_0\in\ell^2$ and $\varepsilon>0$, there is a $N_0(x_0,\varepsilon)$ such that

$$\begin{aligned}\|\Pi_N x(t)\|_{\ell^2}&\leqslant\|\Pi_N e^{-t\tilde{A}}x_0\|_{\ell^2}\\ &\quad+\left\|\int_0^t\omega^2(\tau)\Pi_N\left(e^{(\tau-t)\tilde{A}}Bx(\tau)\right)d\tau\right\|_{\ell^2}\\ &<\varepsilon+\int_0^{+\infty}\omega^2(\tau)\,d\tau\sup_{s\in[0,t]}\|\Pi_N e^{-s\tilde{A}}Bx(t-s)\|_{\ell^2}\end{aligned}$$

for all $t\geqslant 0$ and $N\geqslant N_0(x_0,\varepsilon)$. In accordance with (40), to show that $\|\Pi_N x(t)\|_{\ell^2}$ is uniformly small when N is large enough, it suffices to establish

$$\lim_{N\to\infty}\left(\sup_{t\geqslant 0}\sup_{\tau\in[0,t]}\|\Pi_N e^{-\tau\tilde{A}}Bx(t-\tau)\|_{\ell^2}\right)=0. \tag{49}$$

We need the following lemma.

Lemma 2. *Suppose that $C\subset\ell^2$ is compact. Then*

$$\lim_{N\to\infty}\left(\sup_{\tau\geqslant 0}\sup_{\bar{x}\in C}\|\Pi_N e^{-\tau\tilde{A}}\bar{x}\|_{\ell^2}\right)=0. \tag{50}$$

Proof. Owing to Lemma 1, assertion (50) is equivalent to the following:

$$S=\{e^{-\tau\tilde{A}}\bar{x}\mid \bar{x}\in C,\ \tau\geqslant 0\}\text{ is precompact in }\ell^2.$$

Consider a sequence $\{y_n\}_{n=1}^{\infty}\subset S$. Then there is a pair of sequences $\{x_n\}_{n=1}^{\infty}\subset C$ and $\{\tau_n\}_{n=1}^{\infty}\subset\mathbb{R}_+$ such that $y_n=e^{-\tau_n\tilde{A}}x_n$. As C is compact, there is a subsequence $\{x_{n(k)}\}_{k=1}^{\infty}$ that converges to some $x^*\in C$ as $k\to\infty$. Uniform boundedness of the semigroup $\{e^{-\tau\tilde{A}}\mid\tau\geqslant 0\}$ implies

$$\lim_{k\to\infty}\|e^{-\tau_{n(k)}\tilde{A}}x^*-y_{n(k)}\|_{\ell^2}=0. \tag{51}$$

On the other hand, as $\{e^{-\tau\tilde{A}}x^*\mid\tau\geqslant 0\}$ is precompact, there is a convergent subsequence $\{e^{-\tau_{n(k(m))}\tilde{A}}x^*\}_{m=1}^{\infty}$. Then $\{y_{n(k(m))}\}_{m=1}^{\infty}$ is also convergent because of (51). Therefore, S is precompact. □

Let us remark that the operator $B : \ell^2 \to \ell^2$ is compact as its matrix norm is finite:

$$\frac{k}{c}\sum_{n=1}^{\infty}\frac{1}{\lambda_n} < \infty.$$

The convergency is established here by means of estimate (15). The solutions boundedness for (47) together with compactness of B implies, for each solution $x(t)$, there is a compact $C \subset \ell^2$ such that

$$Bx(t) \in C, \quad \forall t \geqslant 0.$$

Then (49) follows from Lemma 2.

We have shown that, for any solution $x(t)$ of (47), $\|\Pi_N x(t)\|_{\ell^2} \to 0$ as $N \to \infty$ uniformly on $t \in \mathbb{R}_+$. Therefore, $\{x(t) \mid t \geqslant 0\}$ is precompact.

4.3. Characterization of the limit set

To conclude the proof that (35) stabilizes (25) with respect to Π, one has to check conditions 4 and 5 of Theorem 1. The closed-loop system (25), (35) on

$$Z = \overline{\{x \in \mathscr{D}(A) \mid \dot{V}(x) = 0\}}$$

takes the following form:

$$\dot{x}(t) = Ax(t), \;\; \omega(t) = 0, \;\; v(x(t)) = 0.$$

By solving the above linear equations, we get

$$\theta(t) = \theta_0,$$

$$\xi_{in}(t) = C_{in}^{(1)} \sin(\sqrt{\lambda_n}ct) + C_{in}^{(2)} \cos(\sqrt{\lambda_n}ct),$$

$$\eta_{in}(t) = C_{in}^{(1)} \cos(\sqrt{\lambda_n}ct) - C_{in}^{(2)} \sin(\sqrt{\lambda_n}ct), \tag{52}$$

where the constants θ_0, $C_{in}^{(j)}$ are defined by the initial conditions. Since the left-hand side of (35) vanishes on Z, then substitution of (52) into (35) yields

$$\frac{c_0\theta_0}{c} = \sum_{n=1}^{\infty}\sum_{i=1}^{k} J_n\sqrt{\lambda_n}(C_{in}^{(1)} \sin(\sqrt{\lambda_n}ct) + C_{in}^{(2)} \cos(\sqrt{\lambda_n}ct)). \tag{53}$$

Assume that the closed-loop system has a semitrajectory on Z defined for $t \geqslant 0$. Then (53) holds for some constants θ_0, $C_{in}^{(j)}$. The system of functions

$$\{1, \sin\sqrt{\lambda_n}\tau, \cos\sqrt{\lambda_n}\tau \mid \tau \geqslant 0, n \in \mathbb{N}\}$$

is linearly independent on $\mathbb{R}_+$ because of the estimate (15) and Theorem 1.2.17 of Krabs (1992). As $\lambda_n \neq 0$ and also $J_n \neq 0$ in (19), relation (53) is satisfied only if

$$\theta_0 = \sum_{i=1}^{k} C_{in}^{(1)} = \sum_{i=1}^{k} C_{in}^{(2)} = 0, \quad \forall n \in \mathbb{N}. \tag{54}$$

It means that the maximal positively invariant subset of Z is contained in

$$Z_0 = \left\{x \in \ell^2 \mid \theta = \omega = \sum_{i=1}^{k}\xi_{in} = \sum_{i=1}^{k}\eta_{in} = 0, \;\; \forall n \in \mathbb{N}\right\}.$$

Now it follows from $\operatorname{Ker}\Pi = Z_0$ that conditions (4) and (5) of Theorem 1 hold.

According to Theorem 2, *the feedback control* (35) *ensures asymptotic stability of the control system* (25) *with respect to* (28). □

Remark. To implement the proposed feedback control, one has to reconstruct the full (infinite dimensional) system state from the outputs which can be technically measured. We do not consider observer design issues in this paper. Observability of a finite dimensional model with two elastic beams has been studied by Kovalev, Zuyev, and Shcherbak (2002). It has been proved that the finite-dimensional body-beams system is observable, provided that one measures the relative displacements of certain points at the beams.

5. Simulation results

Consider the case $k = 2$. In this section, we present the results of numerical simulation for the closed-loop system obtained from (25), (35) by discarding the terms with $n > 5$. We choose the following (dimensionless) values of the parameters

$$l = \rho = c = J_0 = 1, \quad d = 1/2, \;\; h = 3$$

and the initial conditions

$$\theta(0) = \pi, \quad \omega(0) = 0,$$

$$\xi_{in}(0) = \eta_{in}(0) = 0 \quad (i = \overline{1, 2}, \;\; n = \overline{1, 5}). \tag{55}$$

The evolution of state variables is shown by Figs. 2–4 . These illustrations justify that the feedback law (35) is able to stabilize the system even if $\theta(0)$ is not so small.

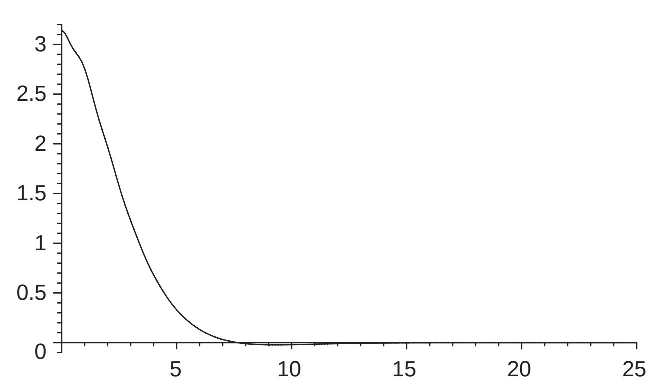

Fig. 2. The closed-loop response $\theta(t)$.

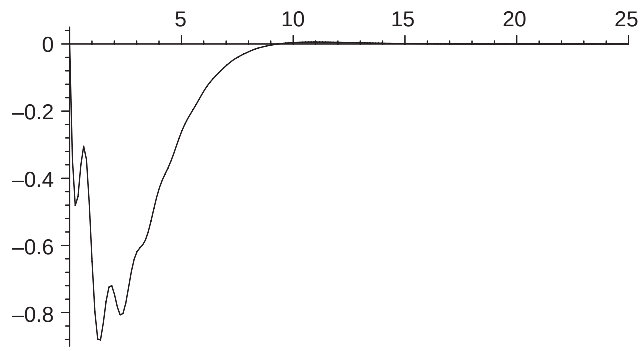

Fig. 3. The closed-loop response $\omega(t)$.

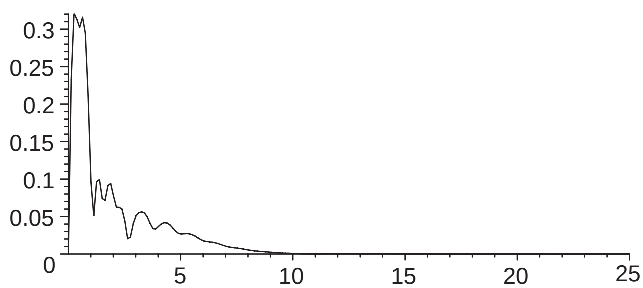

Fig. 4. The closed-loop response $(\sum_{i\leqslant 2}\sum_{n\leqslant 5}(\xi_{in}(t)^2+\eta_{in}(t)^2))^{1/2}$.

Acknowledgements

This work is supported by the Abdus Salam International Centre for Theoretical Physics (Trieste, Italy) and Alexander von Humboldt Foundation (Bonn - Bad Godesberg, Germany). The author expresses sincere gratitude to editors and anonymous reviewers for their valuable comments.

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