CHAPTER TEN

Operator Groups and Unique Decompositions

10A

The subject matter of this chapter is intermediate between the "pure" group theory we have been discussing up to now and module theory, which will constitute an important part of our study of rings. We give definitions and prove theorems that have applications both in group theory and in module theory.

(10.1) DEFINITION. Let X be an arbitrary (possibly empty) set and let G be a group. We say that G is an *X-group* (or *group with operator set X*) provided that for each $x \in X$ and $g \in G$, there is defined an element $g^x \in G$ such that if $g, h \in G$, then $(gh)^x = g^x h^x$.

Essentially, X is just a set of endomorphisms of G, in other words, group homomorphisms from G into itself. We do not require, however, that distinct elements of X determine distinct endomorphisms of G. (In fact, we pay hardly any attention at all to individual elements of X.)

For the purpose of motivating the definition, perhaps the best example of an X-group is a vector space V over some field F. We consider the (additive) group structure of V as an F-group, where the scalar multiplications by the elements of F provide the required endomorphisms. (Note that if $V \neq 0$ in this situation, then distinct elements of F give distinct endomorphisms.)

When we study groups we are usually interested in all subgroups and all homomorphisms, but in vector space theory we limit our attention to subspaces and linear transformations. In general, attaching a set of operators to a group provides a way to focus on a restricted family of subgroups and homomorphisms. (Since every group is an X-group with $X = \varnothing$, however, we can use the language of X-groups even when we do not wish to limit our attention to some privileged collection of subgroups. The results of this chapter, therefore, apply to ordinary groups, too.)

Let G be an X-group. A subgroup $H \subseteq G$ is an X*-subgroup* if for every $x \in X$ and $h \in H$, we have $h^x \in H$. In this situation, we also say that H *admits* X. In the vector space example, the F-subgroups of V are precisely the subspaces of V. It was our custom when discussing pure group theory to have it understood when we wrote "$H \subseteq G$" that H is a subgroup of G and not merely a subset (unless the contrary was clear from the context). Similarly, if G is an X-group and we write "$H \subseteq G$" or "$H \triangleleft G$," our default assumption is that H is an X-subgroup. Note that if G is an X-group and $H \subseteq G$ admits X, then H is automatically an X-group itself, and any X-subgroup of H is an X-subgroup of G.

If G is an arbitrary group, we have seen that we can view G as an operator group simply by taking $X = \varnothing$. In that case, of course, the notions of "subgroup" and "X-subgroup" coincide. We can also view G as an operator group in a different way. Taking $X = G$, we let G act on G by conjugation. Observe that with this construction, the G-subgroups of G are precisely the normal subgroups.

Suppose G and H are X-groups. A homomorphism $\varphi : G \to H$ is an X*-homomorphism* if $\varphi(g^x) = \varphi(g)^x$ for $x \in X$ and $g \in G$. (In the vector space example, of course, the F-homomorphisms between two F-spaces are precisely the linear transformations.) In general, we write $\mathrm{Hom}_X(G, H)$ to denote the set of all X-homomorphisms from G to H.

The results of Chapter 3 on homomorphisms between groups apply without essential change to the situation of X-homomorphisms between X-groups. For instance, kernels and images of X-homomorphisms are X-subgroups, and if $\varphi : G \to H$ is a surjective X-homomorphism, then there is a bijective correspondence between the set of X-subgroups of G containing $\ker(\varphi)$ and the set of all X-subgroups of H. (In fact, the bijection given in the Correspondence theorem (3.7) makes X-subgroups correspond to X-subgroups.)

To make the analogy with our previous work complete, we need to mention that if G is an X-group and $N \triangleleft G$ (and N admits X), then G/N becomes an X-group in a fairly natural way. We define $(Ng)^x = Ng^x$ for $g \in G$ and $x \in X$, and we observe that this is well defined since if $Ng = Nh$, then $h = ng$ for some $n \in N$, and thus $h^x = n^x g^x$. However, $n^x \in N$ because N admits X, and so $Nh^x = Ng^x$.

At this point, the reader will not be surprised to learn that if $\varphi \in \mathrm{Hom}_X(G, H)$ is surjective and $N = \ker(\varphi)$, then G/N and H are X-isomorphic X-groups. (In other words, an isomorphism exists that is, in fact, an X-homomorphism.)

We do not propose to give proofs for all the routine facts about X-subgroups, X-factor groups, and X-homomorphisms we use. Everything works with at most trivial changes from the case where $X = \varnothing$.

10B

Although we have alluded to composition series previously, we have not yet proved any significant theorems concerning them. The major result in this subject is the Jordan-Hölder theorem, which we now intend to prove in the general context of operator groups. By varying our operator set X, we obtain results that are useful in different ways in group theory and we can also get applications to module theory.