Hi,

I'm confused about the exact interpretation of Noether's theorem for fields. I find that the statement of the theorem and its proof are not presented in a precise manner in books.

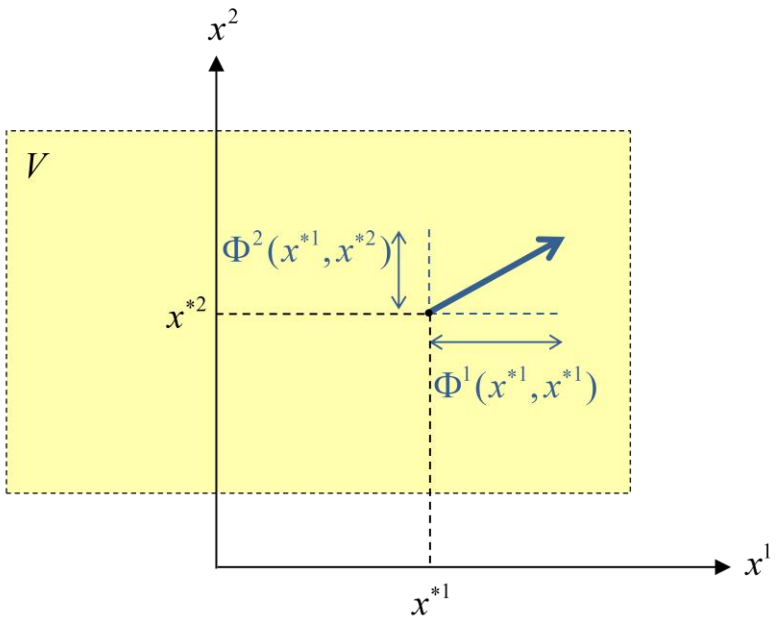
My main question is: what is the precise heuristic argument that leads to Noether theorem?

I'll aid this discussion with some illustrations.

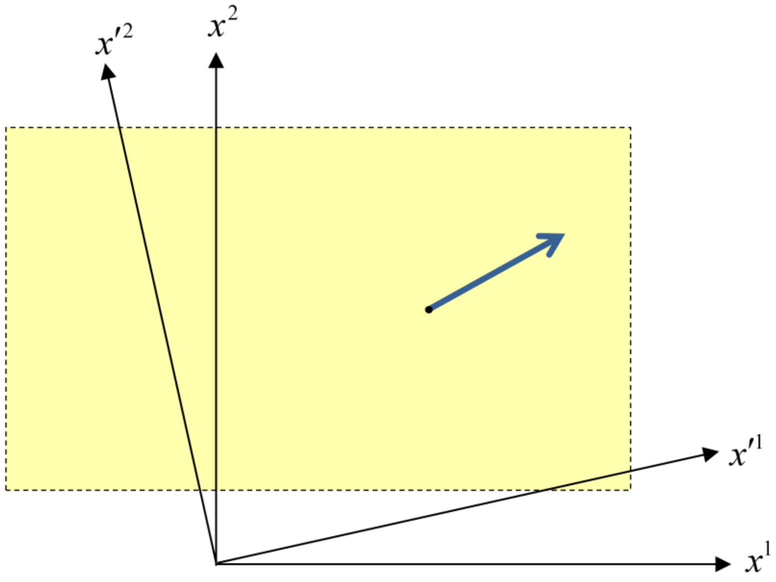
The guidelines for the theorem's proof are: 1) Assume that the action integral is invariant under a certain continuous set of space-time transformations; 2) Perform an infinitesimal transformation on the space-time coordinates; 3) Perform also the resulting infinitesimal transformation on the field components; 4) Equate the action integral of the transformed space-time and field with the action integral of untransformed space-time and field; 5) From the equation, derive Noether's current.

My problem is interpreting step 4, in which two integrals are equated. In the following discussion, I'll use as an example 2-dimentional space, and the field is a 2-vector field.

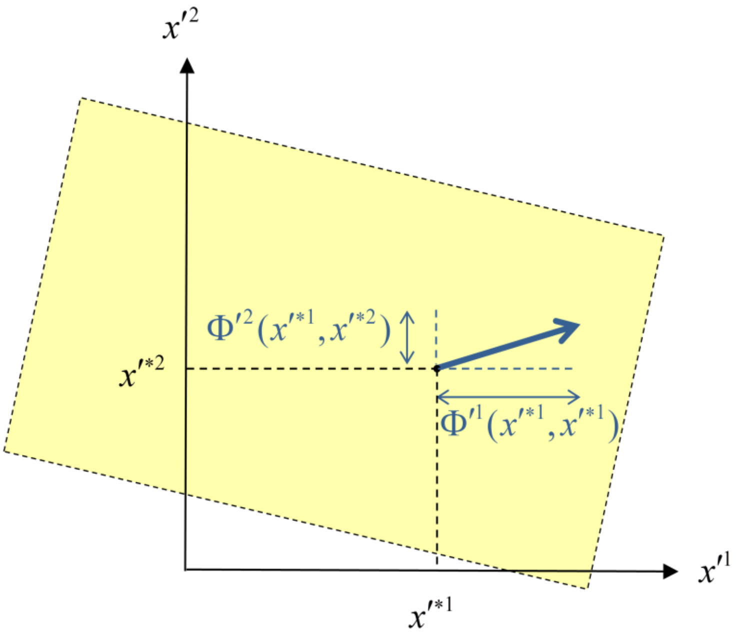
The following figure shows the field Φ at a particular point. A specific coordinates system is being used: x^1x^2. The field has some functional dependence on the coordinates $\Phi^i(x^1,x^2)$ $(i=1,2)$. The particular point has coordinates (x^{*1},x^{*2}). The "action integral" will be an integration over the region V of some function of the field.



Now, a space transformation is performed. The transformation will be viewed in the passive sense: It is not the physical system which is transformed, but rather the coordinates are. We are describing the same field with a different coordinate system. Let's suppose that this transformation is an infinitesimal rotation about the origin of $x^1 x^2$. The new set of coordinates will be denoted $x'^1 x'^2$:



Now, for simplicity, the following figure is aligned with the page, and only the new coordinate system is mentioned:



The field is shown in the same space point, which in the new coordinate system has coordinates (x'^{*1}, x'^{*2}). Obviously, the components of the field have changed, and they are denoted now by $\Phi'^i(x'^{*1}, x'^{*2})$. Therefore, the functional dependence of the function $\Phi'^1(\cdot,\cdot)$ is different from functional dependence of the function $\Phi^1(\cdot,\cdot)$, and similarly for the second component.

Let's suppose that the action integral, when expressed in x^1x^2 coordinate system, is of the form

$$I = \int_a^b dx^1 \int_c^d dx^2 \mathcal{L}\left(\Phi^1(x^1,x^2),\Phi^2(x^1,x^2)\right).$$

Which of the following arguments is used in the proof of Noether's theorem?

1) An observer using the $x'^1x'^2$ coordinate system must use the same laws of physics, and the expression for the action integral is such a law of physics. Therefore, this observer uses the exact same formula for the action integral and gets the exact same result:

$$\int_a^b dx'^1 \int_c^d dx'^2 \mathcal{L}\left(\Phi'^1(x'^1,x'^2),\Phi'^2(x'^1,x'^2)\right) = \int_a^b dx^1 \int_c^d dx^2 \mathcal{L}\left(\Phi^1(x^1,x^2),\Phi^2(x^1,x^2)\right)$$

The function $\mathcal{L}$ has the exact the functional dependence in both sides of the equation (as a function $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$). Notice that two different regions of space are integrated.

2) An observer using the $x'^1x'^2$ coordinate system must use the same laws of physics, and the expression for the action integral is such a law of physics. In the integral, an integration over a space region is performed. Space region is a coordinate-independent notion. Therefore, this observer uses the same formula but adjusts it so that the integration will still be performed on the same space region:

$$\iint_{\text{same space region}} dx'^1 dx'^2 \mathcal{L}\left(\Phi'^1(x'^1,x'^2),\Phi'^2(x'^1,x'^2)\right) = \int_a^b dx^1 \int_c^d dx^2 \mathcal{L}\left(\Phi^1(x^1,x^2),\Phi^2(x^1,x^2)\right)$$

Again, the function $\mathcal{L}$ has the exact the functional dependence in both sides of the equation.

Depending on the answer, I might have further questions to ask regarding the derivation of the Noether current. I'll probably make more use of these figures.