The acceleration of mass *m* on a spring is equal to the spring constant *k* times the displacement *x* from the equilibrium position. We assume initial conditions x(0) = dx/dt (t=0) = 0.

$$m\frac{d^2x}{dt^2} = -kx \qquad [1]$$

Using $\omega_0^2 = k/m$, and using a driving force $F = A_0 \sin(\omega t)$ we get

$$\frac{d^2x}{dt^2} + \omega_0^2 x = A_0 \sin(\omega t) \qquad [2]$$

Assume a particular solution of the form

$$x = a\sin(\omega t) + b\cos(\omega t) \qquad [3]$$

Substitute [3] in [2] and find

$$b = 0, \qquad a = \frac{A_0}{(\omega_0^2 - \omega^2)} \qquad [4]$$

In addition, we have to add the general solution to the equation

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0 \qquad [5]$$

which is

$$x(t) = c\sin(\omega_0 t) + d\cos(\omega_0 t) \qquad [6]$$

So the complete solution to [2] has the form

$$x(t) = c\sin(\omega_0 t) + d\cos(\omega_0 t) + \frac{A_0 \sin(\omega t)}{\omega_0^2 - \omega^2} \qquad [7]$$

Since one initial condition is $x(0) = 0$, then $d = 0$. So we now have for dx/dt

$$\frac{dx}{dt} = c\omega_0 \cos(\omega_0 t) + \frac{A_0 \omega \cos(\omega t)}{(\omega_0^2 - \omega^2)} \qquad [8]$$

Because another initial condition is dx/dt = 0 at t=0, we can solve for c, giving the complete solution for the amplitude of the oscillation:

$$x(t) = \frac{A_0}{(\omega_0^2 - \omega^2)}\left[\sin(\omega t) - \frac{\omega}{\omega_0}\sin(\omega_0 t)\right] \qquad [9]$$

To maximize x(t), $\omega = \omega_0 = \sqrt{k/m}$ or $\omega_0 << \omega$.

Bob S 4/5/12