Question - convolution of two functions

I still can't get the exact $\sin(\omega_0 t)$ result through a detailed computation:

$$g(t) = \cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} G(\omega) = \sqrt{\frac{\pi}{2}}\ \delta(\omega - \omega_0) + \delta(\omega + \omega_0)$$

$$h(t) = \frac{1}{\pi t} \xleftrightarrow{\mathcal{F}} H(\omega) = \frac{j}{\sqrt{2\pi}} \operatorname{sgn}(\omega)$$

then

$$G(\omega)H(\omega) = \sqrt{\frac{\pi}{2}}\ \delta(\omega - \omega_0) + \delta(\omega + \omega_0)\ \cdot \frac{j}{\sqrt{2\pi}} \operatorname{sgn}(\omega) = \frac{j}{2} \operatorname{sgn}(\omega)\ \delta(\omega - \omega_0) + \delta(\omega + \omega_0)$$

$$= \begin{cases} \dfrac{j}{2}\ \delta(\omega - \omega_0) + \delta(\omega + \omega_0)\ , & \omega > 0 \\[2ex] -\dfrac{j}{2}\ \delta(\omega - \omega_0) + \delta(\omega + \omega_0)\ , & \omega < 0 \end{cases}$$

but

$$\mathcal{F}\ \sin(\omega_0 t)\ = j\sqrt{\frac{\pi}{2}}\ \delta(\omega - \omega_0) - \delta(\omega + \omega_0)$$

and I cannot figure out why

$$j\sqrt{\frac{\pi}{2}}\ \delta(\omega - \omega_0) - \delta(\omega + \omega_0) \overset{(?)}{=} \frac{j}{2}\ \delta(\omega - \omega_0) + \delta(\omega + \omega_0)$$

or

$$j\sqrt{\frac{\pi}{2}}\ \delta(\omega - \omega_0) - \delta(\omega + \omega_0) \overset{(?)}{=} -\frac{j}{2}\ \delta(\omega - \omega_0) + \delta(\omega + \omega_0)$$

as the differences are stressed in red.
Why I can't get it right?