SYMMETRY OF A FINITE SEQUENCE OF INTEGERS

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Introduction

For the purposes of this post, a sequence of integers is represented in the x-y Cartesian co-ordinate system in the following form.

$f[x]=1$ if x is a member of the sequence, otherwise

$f[x]=0$, for all x = 0,1,…, 2N.

This discussion is limited to those sequences $S_0, S_1, \ldots S_i, S_j, \ldots S_M$ where $S_i > S_j$ for all $i>j$ and where $S_M <= 2N$. It does not apply to sequences having members where $S_i = S_j$.

A value that is representative of the symmetry of a finite sequence of integers is defined here as

$$I[S]=\frac{\sum_{x=0}^{x=2N} f[x].f[2N-x]}{\sum_{x=0}^{x=2N} f[x].f[x]} \quad \text{for } x = 0,1,\ldots.2N \qquad Eqn(1)$$

As can be seen, the numerator is indicative of the symmetry of the function sequence $f[x]$ about the value N. Only when x and 2N-x are both members of the sequence do they contribute to the value of the symmetry of the sequence. Whereas the denominator is indicative of the power of the function sequence $f[x]$, so that the denominator normalizes the numerator to between zero and one.

The value of $2N$ is chosen somewhat arbitrary, so the value of $I[S]$ measures the symmetry of the sequence S about a chosen value N. A sequence S that is completely symmetrical about N will have a value I[S]=1 whereas a sequence S that is completely asymmetrical will have a value I[S]=0.

As an example consider the sequence $S^{'}$ of all even numbers up to 2N. The function sequence $f[x]=1$ if x is a member of this sequence $S^{'}$, otherwise $f[x]=0$, for all x = 0,1,…, 2N. becomes {(0,0)(1,0),(2,1),(3,0),(4,1)(5,0),(6,1)…..,(2N-1,0), (2N,1)} when expressed in x-y Cartesian co-ordinates. As is apparent, the symmetry value of this function sequence $f[x]$ about N is I[S] = N-1/N. Note the numerator is N-1 and not N because zero is not even number.

Fourier Transform

The value $I[S]$ can also be expressed in the fourier domain, and yields an interesting result.

Using well-established principles relating to multiplication of functions and "time" shifts, the discrete Fourier transform of the function $m[x]=f[x].f[2N-x]$ can be shown as

$$M[k]=\frac{1}{2N}\sum_{l=0}^{2N-1}F[l]F^{*}[k-l]$$

where $F[l]$ is the discrete Fourier transform of the function $f[x]$ namely

$$F[l] \;=\; \sum_{x=0}^{2N-1} f[x]e^{-j2\pi lx/2N} \qquad l=0,1,...2N-1$$

According to well established principles, the DC component of $M[k]$, namely $M[0]$ equals $\sum_{x=0}^{x=2N} f[x].f[2N-x]$.

Also, as $f[x]$ is a real function it is well known that $F[-l]$= $F^*[l]$, and $M[0]$ can then be simplified to

$$M[0] = \frac{1}{2N}\sum_{l=0}^{2N-1} F[l]F^{*}[-l] = \frac{1}{2N}\sum_{l=0}^{2N-1} F^{2}[l]$$

Or in other words

$$\sum_{x=0}^{x=2N} f[x].f[2N-x] \quad = \quad \frac{1}{2N}\sum_{l=0}^{2N-1} F^{2}[l]$$

And from Parseval's theorem it can be seen that

$$\sum_{x=0}^{2N-1} f[x].f[x] = \frac{1}{2N}\sum_{l=0}^{2N-1} F[l].F^{*}[l]$$

and equation (1) becomes

$$I[S] \quad = \quad \frac{\sum_{l=0}^{2N-1} F^{2}[l]}{\sum_{l=0}^{2N-1} F[l].F^{*}[l]}$$

Now according to well established principles

$$F[2N - l] \longleftrightarrow f(-x)e^{-j2\pi l 2N/2N} = f(-x)$$

and as the function f(x) is real then

$$f(-x) \longleftrightarrow F[-l] = F^{*}[l] \quad \text{thus}$$

$$F[2N - l] = F^{*}[l]$$

furthermore as $\mathrm{Re}\{F[0]\} = \pi \quad \mathrm{Im}\{F[0]\} = 0$ and $\mathrm{Re}\{F[N]\} = -\pi \quad \mathrm{Im}\{F[N]\} = 0$

Consequently $\sum_{l=0}^{2N-1} \mathrm{Re}\{F[l]\}.\mathrm{Im}\{F[l]\} \;=\; 0$ and equation (1) becomes

$$I[S] \quad = \quad \frac{\sum_{l=0}^{2N-1} (\mathrm{Re}\{F[l]\})^2 \; - \; (\mathrm{Im}\{F[l]\})^2}{\sum_{l=0}^{2N-1} (\mathrm{Re}\{F[l]\})^2 \; + \; (\mathrm{Im}\{F[l]\})^2}$$

It is interesting to note that the sum of the coefficients $(\mathrm{Re}\{F[l]\})^2$, $(\mathrm{Im}\{F[l]\})^2$ are indicative of the power of the sequence function $f[x]$, whereas the sum of the difference of these coefficients are indicative of the symmetry of the sequence function $f[x]$.

Example—Prime Number Sequence

As an example, the symmetry value is determined for a prime number sequence upto an arbitrarily selected even number 2N. As will be apparent the function $m[x] = f[x].f[2N-x]$ will only be one when both x is a prime and 2N-x is also a prime. This prime couplet (x, 2N-x) is in fact a goldbach partition as it satisfies x + (2N-x) = 2N. Furthermore, the sum $\sum_{x=0}^{x=2N-1} f[x].f[2N-x]$ is equal to the total number of Goldbach partitions for that even number 2*N*. For the purposes of this post the value of Goldbach partitions is designated as g[2N]. It is also apparent that the total number of primes upto 2N, designated here as π[2N] is equal to $\sum_{x=0}^{2N-1} f[x].f[x]$. It thus follows that

$$I[S] \quad = \frac{g[2N]}{\pi[2N]} \quad = \quad \frac{\sum_{l=0}^{2N-1} (\mathrm{Re}\{F[l]\})^2 \; - \; (\mathrm{Im}\{F[l]\})^2}{\sum_{l=0}^{2N-1} (\mathrm{Re}\{F[l]\})^2 \; + \; (\mathrm{Im}\{F[l]\})^2}$$

where $F[l]$ is the fourier transform of the prime number sequence $f[x]$.

As an aside, it can be noted *IF* Goldbach's conjecture is to be shown to be true then the following relationship must hold for all 2N.

$$g[2N] \;=\; \frac{1}{2N} \sum_{l=0}^{2N-1} (\mathrm{Re}\{F[l]\})^2 \; - \; (\mathrm{Im}\{F[l]\})^2 \; > 0$$

where $F[l]$ is the fourier transform of the prime number sequence *f[x].*

viz where $f[x]=1$ if x is a prime number, otherwise

$f[x]=0,$ for all x = 0,1,…, 2N-1.

Returning to the symmetry aspect, it can be seen that the ratio g[2N]/π[2N] viz the number of goldbach partitions divided by the total number of primes is indicative of the symmetry of that prime number sequence upto 2N.

Fig. 1 shown below illustrates the well-known Goldbach's Comet. On the x-axis is listed the even numbers 2N, whereas the y-axis lists the number of Goldbach Partitions g[2N] for that a particular even number 2N. The actual numbers of Goldbach partitions are denoted in blue. The lower line in yellow is the postulated lower bound derived from the Hardy-Littlewood's equation.

[see Wikipedia http://en.wikipedia.org/wiki/Goldbach%27s_conjecture]

$$2\Pi_2 \, (\prod_{p|n;\, p\geq 3} \frac{p-1}{p-2}) \int_2^n \frac{dx}{\ln^2 x} \approx 2\Pi_2 \, (\prod_{p|n;\, p\geq 3} \frac{p-1}{p-2}) \, \frac{n}{\ln^2 n}$$

for simplicity sake we take lower bound of the Goldbach partitions as

$$2\Pi_2 \frac{n}{\ln^2 n}$$

In this post the even number n is represented as $2N$ and this Goldbach lower bound then becomes

$$2\prod_2 \frac{2N}{\ln^2 2N}$$

This conjectured lower bound of Goldbach partitions is shown as a yellow line in Fig. 1. A guess estimate for the upper bound of Goldbach is also shown as a yellow line in Fig. 1. This guess estimate, which appears to be on the high side, is

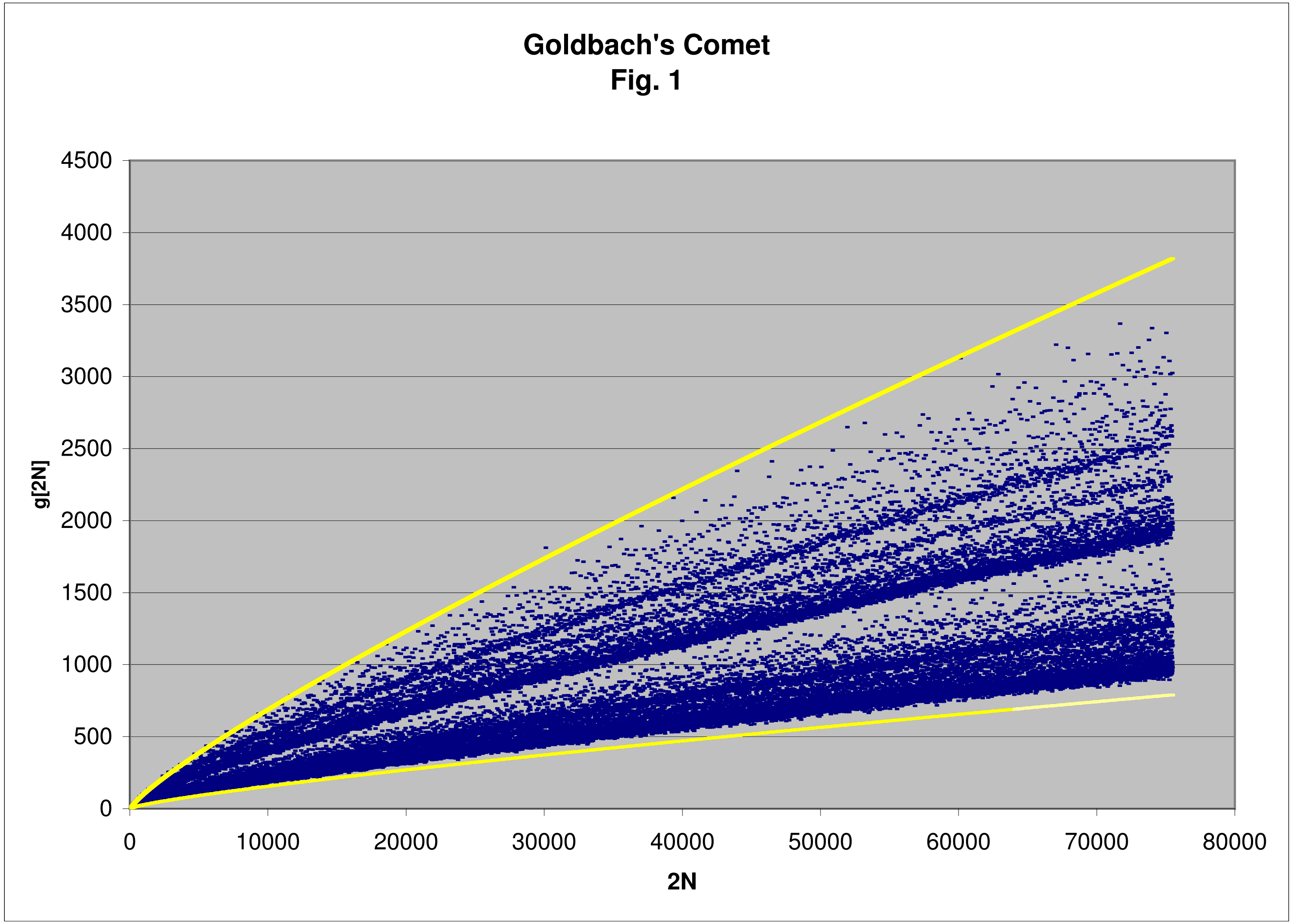
$$4\prod_2 \frac{2N\ln(\ln(2N))}{\ln^2 2N}$$

Fig. 2 shown below illustrates the symmetry of a prime number sequence up to an even number 2N about N. On the x-axis is listed the even numbers 2N, whereas the y-axis lists the Symmetry Value I[S] which in this particular case equals g[2N]/π[2N], where g[2N] is the number of Goldbach Partitions g[2N] for that particular even number 2N, and π[2N] is the total number of primes up to that even number 2N. The actual symmetry values I[S] are denoted in blue. The postulated upper and lower bounds (again shown in yellow in Fig. 2) are derived from Goldbach bounds plus Gauss estimate for the total number of primes upto 2N viz $2N/\ln[2N]$

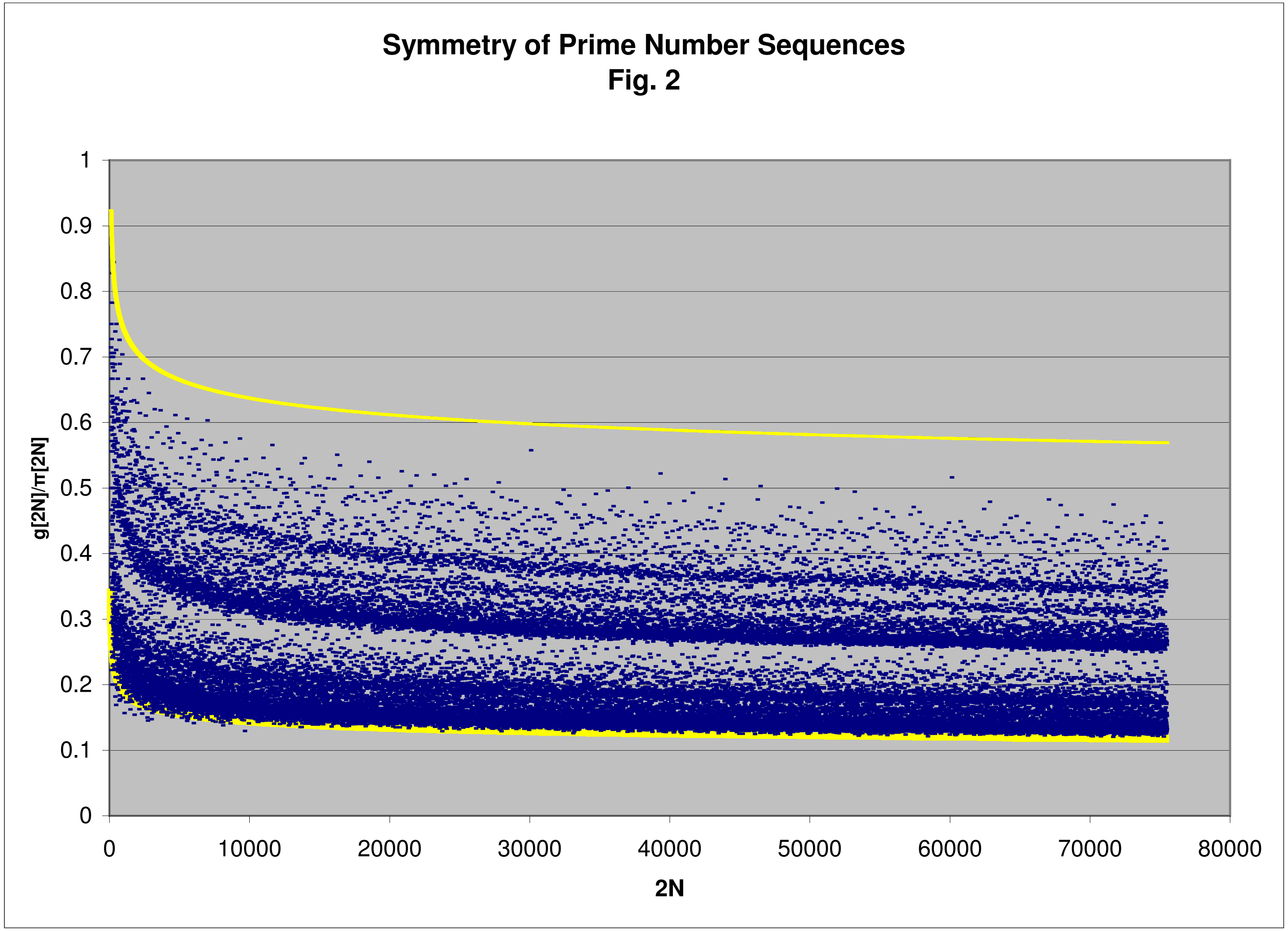
Namely, the conjecture lower bound for g[2N]/π[2N] is then $2\prod_2 \frac{1}{\ln\ 2N}$

and the conjectured upper bound for g[2N]/π[2N] is then $4\prod_2 \frac{\ln(\ln(2N))}{\ln 2N}$

As can be seen the symmetry value g[2N]/π[2N] appears to approach zero as 2N -> infinity, and as such the prime number sequences generally becomes more asymmetric as 2N approaches infinity.



Goldbach's Comet
Fig. 1
4500
4000
3500
3000
2500
2000
1500
1000
500
0
g[2N]
0
10000
20000
30000
40000
50000
60000
70000
80000
2N


Symmetry of Prime Number Sequences
Fig. 2
g[2N]/π[2N]
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10000
20000
30000
40000
50000
60000
70000
80000
2N

References:

1. Principles of Communications, Zeimer and Tranter, c 1976

2. "Some Problems of "Partitio Numerorum" (III). On the expression of a number as a sum of primes." by G.H. Hardy and J.E. Littlewood, ACTA MATHEMATICA, Vol 44. (1922) pp 1-70.