Sample solutions to Homework 2

Q.1

(i) If we assume that p_n is the amount of digoxin present at n-th day ($n = 0, 1, 2, \cdots$), then $p_{n+1} - p_n = kp_n$ because of the change in concentration per day is proportional to the amount of digoxin present. Here k is some constant. In Figure 1, we find that the assumption is correct since $k \approx -0.3101$.

Figure 1: The change in concentration per day versus the amount present. $k \approx -0.3101$.

(ii) The refined model is $p_{n+1} - p_n = kp_n + 0.1$.

(iii) The table of values from the model of (ii) for 15 days is as below:

t	0	1	2	3	4	5	6	7
y	0.5000	0.4449	0.4070	0.3808	0.3627	0.3502	0.3416	0.3357
t	8	9	10	11	12	13	14	15
y	0.3316	0.3288	0.3268	0.3255	0.3245	0.3239	0.3235	0.3232

Q.2

Let $x(t)$ be the amount of the pollutant in the reservoir at time t. The following mathematical model can be established

$$\frac{dx(t)}{dt} = 5\times 10^8 \times 0.05\times 10^{-2} - 5\times 10^8 \times \frac{x(t)}{8\times 10^9},\ t>0$$
$$x(0) = 0.25\times 10^{-2}\times 8\times 10^9 = 2\times 10^{-7}.$$

Solving the equations, we get

$$x(t) = 1.6\times 10^7 e^{-0.0625t} + 4\times 10^6.$$

Let $x(t) = 0.1\times 10^{-2}\times 8\times 10^9$, we obtain $t = \ln(4)/0.0625 \approx 22.1807$. Thus it will take about 23 days to reduce the pollutant concentration in the reservoir to 0.1%.

Q.3

(i) Solving the mathematical model for social diffusion

$$\frac{dX(t)}{dt} = kX(N-X), \quad t>0,$$

we obtain the solution:

$$X(t) = \frac{NX_0e^{kNt}}{N-X_0+e^{kNt}}$$

when $X(0) = X_0$. Figure 2 shows that it leads to a logistic curve when k=0.01 and N=1000.

(ii)When $kX(N-X)$ reaches its maximum, we get $X(t) = N/2$. From $\frac{Ne^{kNt}}{N-1+e^{kNt}} = N/2$, we obtain that the information spread fastest when $t = \ln(N/X_0 - 1)/(kN)$.

(iii) From the solution $X(t) = \frac{NX_0e^{kNt}}{N-X_0+e^{kNt}}$, we know that $X(t)\to N$ when $t\to\infty$. All people will will eventually receive the information.

Figure 2: A plot for X(t) at k=0.01 and N=1000.

(iv) From $f(X) := kX(N - X) = 0$, we know that there are two equilibrium points, one is $X = 0$ and the other is $X = N$. Because of $f'(X) = kN - 2kX$, we know that $f'(0) = kN > 0$ and $f'(N) = -kN < 0$. Therefore at the point $X = 0$, it is unstable; at the point $X = N$, it is stable.

Q.4

Solving the ode $\frac{dN(t)}{dt} = aN - bN^2$ with $N(0) = N_0$, we get the solution $N(t) = \frac{aCe^{at}}{1+bCe^{at}}$ with $C = N_0/(a - N_0 b)$. When $aN(0) = 8, bN(0) = 6$ with $N(0) = N_0 = 120$, we obtain $a = 1/15$, $b = 1/2400$, and $C = 7200$. In this case the solution is $N(t) = \frac{480e^{t/15}}{1+3e^{t/15}}$ and the limiting population $K = 8 \times 120/6 = 160$. Therefore from $0.95 \times 160 = N(t)$, we get the time $t = 15\ln(19/3) \approx 27.6874$. It will take about 28 months for $N(t)$ to reach 95% of the limiting population K.

Q.5

Let $v(t)$ be the velocity of the bomb, the model for $v(t)$

$$\frac{dv}{dt} = g - kv^2, \ t > 0$$

$$v(0) = 0$$

where g is the acceleration of the gravity and k is some positive constant. Solving the model for $v(t)$, we get $v(t) = \frac{\sqrt{g}\left(e^{2\sqrt{gk}t}-1\right)}{\sqrt{k}\left(e^{2\sqrt{gk}t}+1\right)}$.

Let $s(t)$ be the displacement of the bomb after dropped from the plane, then

$$\begin{aligned} s(t) &= \int_0^t \frac{ds}{dt}dt = \int_0^t v(t)dt, \\ &= \int_0^t \frac{\sqrt{g}\left(e^{2\sqrt{gk}t}-1\right)}{\sqrt{k}\left(e^{2\sqrt{gk}t}+1\right)}dt, \\ &= -\sqrt{\frac{g}{k}}t + \frac{1}{k}\ln\left[\left(e^{2\sqrt{gk}t}+1\right)/2\right]. \end{aligned}$$

From $v(t) = \frac{\sqrt{g}\left(e^{2\sqrt{gk}t}-1\right)}{\sqrt{k}\left(e^{2\sqrt{gk}t}+1\right)}$, we find that $v(t) = \frac{\sqrt{g}\left(1-e^{-2\sqrt{gk}t}\right)}{\sqrt{k}\left(1+e^{-2\sqrt{gk}t}\right)}$. When $t \to \infty$, $v(t) \to \sqrt{\frac{g}{k}} = 400\ km/h = 1000/9\ m/s^2$. Thus $k = 7.9461 \times 10^{-4} m^{-1}$ because g=9.81m/s^2.

To get the time delay, we know that $s(t) = 10000m - 600m = 9400m$. Thus from the equation

$$-\sqrt{\frac{g}{k}}t + \frac{1}{k}\ln\left[\left(e^{2\sqrt{gk}t}+1\right)/2\right] = 9400,$$

we find that

$$e^{9400k+\sqrt{gk}t} = 0.5e^{2\sqrt{gk}t} + 0.5.$$

If we let $x = e^{\sqrt{gk}t}$ and $A = e^{9400k} \approx 1753.4$, we then get

$$x^2 - 2Ax + 1 = 0,$$

from which we obtain $x \approx 2 \times 1753.4$. Thus we get $t \approx 92.4506$ from $e^{\sqrt{gk}t} \approx 3506.8$,

Therefore we should set the time delay to be 92.4506 seconds.

Q.6

(i) Let us choose t as the time variable, $v = v(t)$ as the velocity of the rocket, and $m = m(t)$ as the mass of the rocket, we can obtain the following mathematical model

$$m\frac{dv}{dt} - (-40000)\frac{dm}{dt} = -mg - 100 \times 2v$$

before all the fuel is consumed. When $t \leq 250$, the mass of the rocket satisfies

$$\frac{dm}{dt} = -100, m(0) = 25000,$$

from which we obtain $m = m(t) = 25000 - 100t$ for $0 \leq t \leq 250$. Plugging it into the model, we can easily obtain the initial value problem given in the question.

(ii) We first change the equation into the form

$$\frac{dv}{dt} = \left(\frac{40000}{250 - t} - 9.8\right) - \frac{2}{250 - t}v.$$

Integrating the equation, we get the solution

$$v(t) = 20000 - 9.8(250 - t) + C(250 - t)^2.$$

Because of $v(0) = 0$, we obtain $C = -0.2808$ by plugging it into the above solution. Hence the velocity of bomb is

$$v(t) = 20000 - 9.8(250 - t) - 0.2808(250 - t)^2. \tag{0.1}$$

(iii) when t=20000/100=200, i.e., the time when all the fuel had been consumed, we obtain the velocity $v(t) = 18808$ (m/s) from the formula that is given in (ii). After all the fuel have been consumed, the velocity of the rocket is extremely high and the rocket will move upward until it reaches its maximum height.

Q.7

(i)Integrating the equation $\frac{dC(t)}{e^C} = -kdt$ over [0,T], we get the solution

$$C(T) = -\ln\left(e^{-Q} + kT\right)$$

because of $C(0) = Q$. Therefore we know the residual

$$R_1 = -\ln\left(e^{-Q} + kT\right)$$

remains in the blood after T hr has elapsed.

(ii)Integrating the equation $\frac{dC(t)}{e^C} = -kdt$ over [T,2T], we get the solution

$$C(2T) = -\ln\left(kT\left(1 + e^{-Q}\right) + e^{-2Q}\right)$$

because of $C(T) = R_1 + Q$. We then get the residual

$$R_2 = -\ln\left(kT\left(1 + e^{-Q}\right) + e^{-2Q}\right)$$

after the second dose and T hr have elapsed again.

(iii) Integrating the equation $\frac{dC(t)}{e^C} = -kdt$ over $[2T, 3T]$, we can obtain $C(3T) = -\ln\left(kT\left(1 + e^{-Q} + e^{-2Q}\right) + e^{-3Q}\right)$.

Doing the above precedure repeatedly, we can obtain

$$\begin{aligned} C(nT) &= -\ln\left(kT\left(1 + e^{-Q} + e^{-2Q} + \cdots + e^{-(n-1)Q}\right) + e^{-nQ}\right), \\ &= -\ln\left(kT\frac{1 - e^{-nQ}}{1 - e^{-Q}} + e^{-nQ}\right). \end{aligned}$$

Hence the limiting value $R = \lim_{n\to\infty} C(nT) = -\ln\left(\frac{kT}{1-e^{-Q}}\right)$.

(iv) Because $\frac{dC(t)}{dt} = -ke^C < 0$, we have

$$\begin{aligned} & L \le R_0 := Q \le H, \\ & L \le R_1 < Q + R_0 \le H, \\ & L \le R_2 < Q + R_1 \le H, \\ & L \le R_3 < Q + R_2 \le H, \\ & \vdots \\ & L \le R_n < Q + R_{n-1} \le H, \\ & \vdots \end{aligned}$$

Thus we can obtain $L \le Q \le H$ and $R + Q \le H$ (i.e., $-\ln\left(\frac{kT}{1-e^{-Q}}\right) + Q \le H$).

From $Q + R_0 = 2Q \leq H$ and $L \leq Q \leq H$, we obtain $2L \leq H$, i.e., $H - L \geq L$. Hence we get $L \leq H - L \leq H$. By setting $Q = H - L$ in the inequality $-\ln\left(\frac{kT}{1-e^{-Q}}\right) + Q \leq H$, we can find that

$$kT \geq e^{-L} - e^{-H}.$$

On the other hand, we can find $-\ln\left(e^{-Q} + kT\right) \geq L$ because $L \leq R_1$. Thus we get

$$kT + e^{-Q} \leq e^{-L},$$

$$kT \leq e^{-L} - e^{-Q},$$

and

$$kT \leq e^{-L} - e^{-H}$$

because of $Q \leq H$.

Finally we know that

$$kT = e^{-L} - e^{-H}.$$

The proof is completed.