

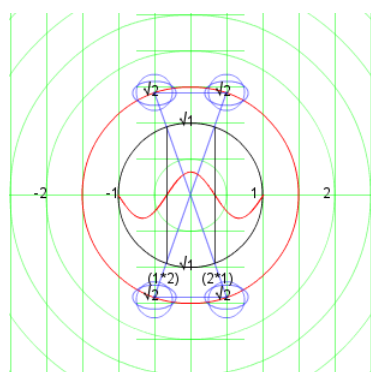
As you increment  $d$  from 1, 2, 3, ... onwards, you do two things:

- Draw concentric circles, whose radii begin at 1 and increment by  $\frac{1}{2}$ . In other words, on each turn the circle's diameter equals  $d + 1$ .
- Draw a sinusoid inside the unit circle, with increasing frequency.

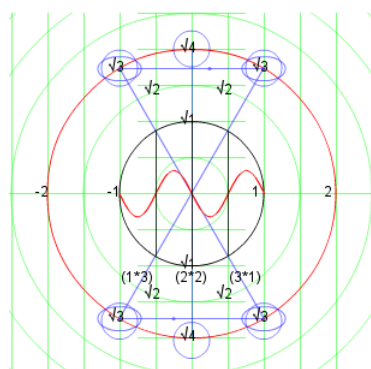
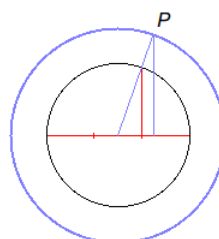
Now, please notice that you are not really using anything from the sinusoid, other than its zero ( $x$ -axis) intersects. That is, the sinusoid is a pretty artifact used **only** to split the diameter of the unit circle ( $= 2$ ) in  $d + 1$  equal parts, each of length  $\frac{2}{d+1}$ .

Of these equally spaced marks on the unit circle, the next-to-last is chosen (the one which is one space from the right end of the unit circle), in order to draw a vertical line from it. Since this mark is at a distance of  $\frac{2}{d+1}$  (the space between marks) from the right end of the diameter (which is at  $x = 1$ ), the mark (and the vertical line) has  $x$  coordinate  $= 1 - \frac{2}{d+1}$ .

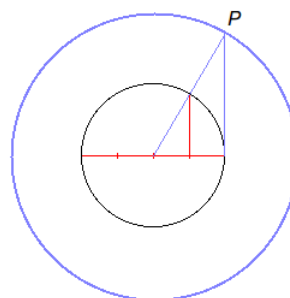
Here are a few screenshots of your Flash application for the cases  $d = 2$  and  $d = 3$ , together with a simplified diagram of the distances involved.



Unit diameter divided in 3 parts



Unit diameter divided in 4 parts



In each case, the intriguing feature is that the height of the point marked as  $P$  on the outer circle has a height ( $y$ -coordinate) which appears to be exactly the square root of  $d$ . Here is an explanation of why this occurs.

The simplified figures on the right show two right triangles, sharing a vertex (at the origin) and the angle at that vertex. For a circle of radius  $R$ , any point of the

circle has coordinates

$$x = R \cos(\text{angle})$$

$$y = R \sin(\text{angle})$$

and, as the *angle* progresses, the point traces the circle. On the unit circle (where  $R = 1$ ), the coordinates are simply the cosine and sine of the sweeping angle.

We know the  $x$ -coordinate of the point on the inner circle ( $x = 1 - \frac{2}{d+1}$ ); thus the *angle* is the arc-cosine of that value of  $x$ .

Now, the outer circle has diameter  $d + 1$  and therefore radius  $\frac{d+1}{2}$ , and so the height of  $P$  is

$$\begin{aligned} P_y &= \left( \frac{d+1}{2} \right) \sin(\text{angle}) \\ &= \left( \frac{d+1}{2} \right) \sin \left( \arccos \left( 1 - \frac{2}{d+1} \right) \right) \end{aligned}$$

and, since

$$\sin(\arccos(x)) = \sqrt{1 - x^2}$$

the height of  $P$  simplifies to

$$\begin{aligned} P_y &= \left( \frac{d+1}{2} \right) \sqrt{1 - \left( 1 - \frac{2}{d+1} \right)^2} \\ &= \left( \frac{d+1}{2} \right) \sqrt{1 - \left( 1^2 - 2 \frac{2}{d+1} + \frac{2^2}{(d+1)^2} \right)} \\ &= \left( \frac{d+1}{2} \right) \sqrt{\frac{4}{d+1} - \frac{4}{(d+1)^2}} \\ &= \sqrt{\frac{4(d+1)^2}{4(d+1)} - \frac{4(d+1)^2}{4(d+1)^2}} \\ &= \sqrt{d+1 - 1} = \sqrt{d} \end{aligned}$$

as expected.

Please note that the formula for the height of  $P$  is **not**  $\tan(\arccos(\dots))$ ; the tangent of the angle is represented, on the unit circle, by a line (drawn below in green) departing from the exact end of the diameter (and, appropriately, tangent to the unit circle); this vertical line only coincides with  $P$  in the case where the outer circle has radius 2 (because  $\tan(\arccos(\frac{1}{2}))$  happens to be  $\sqrt{3}$ ), but not in the general case.

