8330 Direct Limits 2/14/2002

A direct limit is a generalization of a direct sum. It can be defined in great generality: i.e. just as direct sum can be defined in general as a coproduct, a direct limit can be defined as a coproduct with amalgamation. Let us distinguish however between a very general definition of a direct limit, and an actual construction of one. A definition in this general sense is merely a list of properties that a direct limit should have. The question of whether anything exists having those properties in each setting is entirely another matter. That is to say, it is possible to define things which turn out not to exist. A construction of a direct limit on the other hand actually produces one.

Some books "define" a direct limit by giving a construction of a direct limit without proving it satisfies the desired properties, while others give the definition by stating the properties without proving an object having those properties actually exists. We want to do both. The appendix to Kempf for instance constructs rather than defines a direct limit. Indeed he never actually gives the definition via the universal mapping property, as Paulo did in class. Kempf's statement at the bottom of page 143 that "usually one can define a direct limit in a category", should be read as "direct limits do exist in most categories we will encounter". In general I prefer to give the abstract definition and then give one or more constructions, indicating why they have the desired properties.

Recall that a direct sum of an indexed collection of abelian groups $\{M_j\}$ is an abelian group M together with homomorphisms $f_j:M_j \to M$, such that any collection of homomorphisms $g_j:M_j \to X$ into any abelian group X, factor through a unique map $M \to X$. I.e. given the family $\{g_j\}$, there is a unique map $g:M \to X$ such that for every j we have $g_j = g \circ f_j$. Thus maps out of M are equivalent to maps out of the indexed family $\{M_j\}$, and the equivalence is established via the one family $\{f_j\}$. This one family is called a universal such family for this reason. Thus given an indexed family of abelian groups, a direct sum for them is a universal family of maps out of them, but more often one calls the target for these maps, namely M, the direct sum.

The more important role however is played by the universal maps f_j into M. It is by means of these maps that M acquires the structure of a direct sum of the $\{M_j\}$. I.e. the key feature is the way the groups M_j are mapped into M. It follows in particular that any family of maps $h_j:M_j \to N_j$ between two indexed families $\{M_j\} \to \{N_j\}$ induces a unique map $M \to N$ of their direct sums. This map preserves identities and compositions, i.e. the direct sum construction is a functor from indexed families of abelian groups to single abelian groups, and from indexed families of homomorphisms to single homomorphisms.

A direct limit of abelian groups starts also from an indexed family $\{M_j\}$ only this time there are connections between the groups M_j provided by some given maps $h_{jk}:M_j \to M_k$ between certain pairs of the groups, and we require the universal family of maps to respect these connections. More precisely, we are interested now in families of maps $g_j:M_j \to X$ out of the $\{M_j\}$ such that whenever there is a map $M_j \to M_k$, the map g_j should equal the composition $g_k \circ h_{jk}:M_j \to M_k \to X$. A direct limit would be a universal family $\{f_j\}$ of such compatible maps, if one exists.

"Abstract nonsense":

Before going into the simpler case we will use, we continue with the general discussion a bit further, just for fun, and to see the general version of a categorical direct limit once in our lives. This is probably not too useful, but may be good for impressing people at parties, or on oral prelims. Let I,C be any categories and F:I-->C a functor. We will define the direct limit of F. Given any object X in C, define a constant functor c_X:I-->C taking every object in I to X, and every map in I to the identity map of X. This defines a functor c from C to the category Fun(I,C), taking each object X to the corresponding constant functor c_X.

Then a direct limit construction in this setting is an "adjoint functor" for c. I.e. a functor lim:Fun(I,C)-->C, and an isomorphism of bifunctors $\mathrm{Hom}_{\mathrm{func}}(* , c_{(\#)}) \approx \mathrm{Hom}_C(\lim(*), \#)$. In particular if F:I-->C is a functor, a direct limit for F is an object lim(F) in C together with an isomorphism $\mathrm{Hom}_{\mathrm{func}}(F, c_X) \approx \mathrm{Hom}_C(\lim(F),X)$ for every X, which is natural in X. But the construction is also functorial in F. I.e. if X is any object, we have isomorphisms $\mathrm{Hom}_{\mathrm{func}}(F, c_X) \approx \mathrm{Hom}_C(\lim(F),X)$ are also natural in F. I.e. a natural transformation F-->G induces a map $\mathrm{Hom}_{\mathrm{func}}(G, c_X)$-->$\mathrm{Hom}_{\mathrm{func}}(F, c_X)$ as well as a map lim(F)-->lim(G)such that the compositions $\mathrm{Hom}_{\mathrm{func}}(G, c_X)$-->$\mathrm{Hom}_{\mathrm{func}}(F, c_X) \approx \mathrm{Hom}_C(\lim(F),X)$ and
$\mathrm{Hom}_{\mathrm{func}}(G, c_X) \approx \mathrm{Hom}_C(\lim(G), X)$ --> $\mathrm{Hom}_C(\lim(F),X)$ are equal.

So a direct limit construction is a way to transform connected familes of objects into single objects, and maps between connected families of objects into maps between single objects. I.e. a functor F:I-->C is a family of objects in C indexed by the elements of the category I, and connected by the maps F assigns to the maps from I, and lim:Fun(I,C)-->C transforms such familes into objects in C. The elements of $\mathrm{Hom}_{\mathrm{func}}(F, c_X)$ are the "families of maps" from the family of objects indexed by F into the constant family X, and the elements of $\mathrm{Hom}_C(\lim(F),X)$ are the single maps they correspond to. The "universal" family of maps is the family in $\mathrm{Hom}_{\mathrm{func}}(F, c_{(\lim(F))})$ corresponding to the identity map in $\mathrm{Hom}_C(\lim(F),\lim(F))$.
In the situation above of a direct sum, the category I is the index set, there are no maps except identities in this category, the category C is abelian groups, and a functor F:I-->C is merely an indexed collection of abelian groups $\{M_j\}$. A map from this functor to the constant functor c_X is merely an indexed collection of maps M_j-->X.

Limits of directed systems of abelian groups
The generalization from direct sums to direct limits takes for I a directed set, i.e. a set with a partial order, such that for every pair of indices j,k there exists some r with both j≥r and k≥r. (This set is directed downwards, like Kempf's but they could as well be directed upwards. For example the natural numbers are directed upwards.) To view I as a category we consider there is one map in I from j to k if and only if j≥k. Then a functor from I to Ab (abelian groups), is simply an indexed collection of abelian groups $\{M_j\}$, plus a map $h_{jk}:M_j$-->M_k whenever j≥k, and such that h_{jj} = the identity on M_j, and such that $h_{jr} = h_{kr} \circ h_{jk}$ whenever $j \geq k \geq r$. Then, if X is any abelian group, a directed system of maps $\{M_j\}$-->X is a family of maps $g_j:M_j$-->X such that whenever j≥k, the map g_j should equal the composition $g_k \circ h_{jk}:M_j$-->M_k-->X. A direct limit of the system $\{M_j\}$ is a universal family $\{f_j\}$ of such compatible maps. I.e. it is a directed

system of maps $\{f_j:M_j\text{-->}M\}$ such that whenever $\{g_j:M_j\text{-->}X\}$ is a directed system of maps, there exists a unique map g:M-->X such that $g o f_j = g_j$ for every j.