If we have an Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}(t)$, the time evolution operator (taking for simplicity $t_0 = 0$) in the interaction picture is:

$$\hat{U}_I = Id + \sum_{j=1}^{+\infty} \frac{1}{(i\hbar)^j} \prod_{k=1}^{j} \int_0^{t(k)} dt^{(k+1)}\, \hat{V}_I(t^{(k+1)}) \tag{1}$$

con $t(1) = t$. In the Schrödinger picture the time evolution operator turns out to be:

$$\hat{U}_S = e^{\frac{\hat{H}_0 t}{i\hbar}}\, \hat{U}_I = e^{\frac{\hat{H}_0 t}{i\hbar}} + \sum_{j=1}^{+\infty} \frac{1}{(i\hbar)^j}\, e^{\frac{\hat{H}_0 t}{i\hbar}} \prod_{k=1}^{j} \int_0^{t(k)} dt^{(k+1)}\, \hat{V}_I(t^{(k+1)}) \tag{2}$$

The first term is just the evolution operator for the unperturbed Hamiltonian. Then I'll focus on the other terms. I'll suppose that $\hat{H}_0$ has this resolution of the identity (non degenerate eigenvalues):

$$Id = \sum_n |n\rangle \langle n| + \int dE\, |E\rangle \langle E| \tag{3}$$

Then one has:

$$\begin{aligned}
&\frac{1}{(i\hbar)^j}\, e^{\frac{\hat{H}_0 t}{i\hbar}} \prod_{k=1}^{j} \int_0^{t(k)} dt^{(k+1)}\, \hat{V}_I(t^{(k+1)}) = \\
&\frac{1}{(i\hbar)^j} \sum_n \sum_m \left\{ \langle n|\, e^{\frac{\hat{H}_0 t}{i\hbar}} \prod_{k=1}^{j} \int_0^{t(k)} dt^{(k+1)}\, \hat{V}_I(t^{(k+1)})\, |m\rangle \right\} |n\rangle \langle m| + \\
&\frac{1}{(i\hbar)^j} \int dE \int dE' \left\{ \langle E|\, e^{\frac{\hat{H}_0 t}{i\hbar}} \prod_{k=1}^{j} \int_0^{t(k)} dt^{(k+1)}\, \hat{V}_I(t^{(k+1)})\, |E'\rangle \right\} |E\rangle \langle E'|
\end{aligned} \tag{4}$$

and this is equal to:

$$\begin{aligned}
&\frac{1}{(i\hbar)^j} \sum_n \sum_m \left\{ e^{-i\omega_n t}\, \langle n| \prod_{k=1}^{j} \int_0^{t(k)} dt^{(k+1)}\, \hat{V}_I(t^{(k+1)})\, |m\rangle \right\} |n\rangle \langle m| + \\
&\frac{1}{(i\hbar)^j} \int dE \int dE' \left\{ e^{-i\omega(E)t}\, \langle E| \prod_{k=1}^{j} \int_0^{t(k)} dt^{(k+1)}\, \hat{V}_I(t^{(k+1)})\, |E'\rangle \right\} |E\rangle \langle E'|
\end{aligned} \tag{5}$$

Now the quantity between curly brackets is a complex number that depends on t: its expression in the Schrödinger picture is awful but it is still a complex number! I'll call it $c_j^{(n,m)}(t)$ in the discrete case and $c_j(t, E, E')$ in the continuous case. For simplicity I have put also the factor $\frac{1}{(i\hbar)^j}$ into the curly brackets. Then I have:

$$\begin{aligned}
\hat{U}_S = e^{\frac{\hat{H}_0 t}{i\hbar}} &+ \sum_{j=1}^{+\infty} \sum_n \sum_m \left\{ c_j^{(n,m)}(t) \right\} |n\rangle \langle m| + \\
&\sum_{j=1}^{+\infty} \int dE \int dE' \left\{ c_j(t, E, E') \right\} |E\rangle \langle E'|
\end{aligned} \tag{6}$$

Now I'll take a bound state $|i\rangle$ of the unperturbed Hamiltonian $\hat{H}_0$ as initial state, I'll calculate $|i_t\rangle$ and then I'll calculate $\langle\epsilon|i_t\rangle$ where $|\epsilon\rangle$ is a continuous state of the unperturbed Hamiltonian $\hat{H}_0$. I have:

$$\begin{aligned}&\langle\epsilon|i_t\rangle = \langle\epsilon|\hat{U}_S|i\rangle = \langle\epsilon|e^{\frac{\hat{H}_0 t}{i\hbar}}|i\rangle + \sum_{j=1}^{+\infty}\sum_n\sum_m \left\{c_j^{(n,m)}(t)\right\}\langle\epsilon|n\rangle\langle m|i\rangle + \\ &\sum_{j=1}^{+\infty}\int dE\int dE'\left\{c_j(t,E,E')\right\}\langle\epsilon|E\rangle\langle E'|i\rangle\end{aligned} \tag{7}$$

Now $\langle m|i\rangle = \delta_{m,i}$ and $\langle\epsilon|E\rangle = \delta(\epsilon - E)$ so i obtain:

$$\begin{aligned}&\langle\epsilon|i_t\rangle = e^{-i\omega_i t}\langle\epsilon|i\rangle + \sum_{j=1}^{+\infty}\sum_n \left\{c_j^{(n,i)}(t)\right\}\langle\epsilon|n\rangle + \\ &\sum_{j=1}^{+\infty}\int dE'\left\{c_j(t,\epsilon,E')\right\}\langle E'|i\rangle\end{aligned} \tag{8}$$

Now if we have that $\langle E|n\rangle = \langle n|E\rangle = 0$ as you said on the thread, it seems that I obtain $\langle\epsilon|i_t\rangle = 0$ because all of the terms in the sum running on j are 0 as the first term is.