[*The basis for the following work will be the definition of the trigonometric functions as ratios of the sides of a triangle inscribed in a circle; in particular, the sine of an angle will be defined to be the ratio of the triangle's opposite side to its hypotenuse - which here simplifies to being the triangle's opposite side, as noted in the work. I make no reference to the series definitions of the trigonometric functions. Angles are given in radian measure.*

The explicit statement of the theorem proffered as data is given by Euclid VI 33.]

Continuity and Differentiability of the Trigonometric Functions

It may be taken as data that the ratio of the area of a circle to that (A_s) of a sector therein is equal to the ratio of the circumference of the circle to the length ($\theta_s r$) of the sector's arc; and thus

$$\frac{A_s}{\pi r^2} = \frac{\theta_s r}{2\pi r} \qquad \Rightarrow \qquad A_s = \frac{\theta_s r^2}{2} \tag{1}$$

Below is a geometric figure, illustrating a comparison of areas (which will prove useful.)

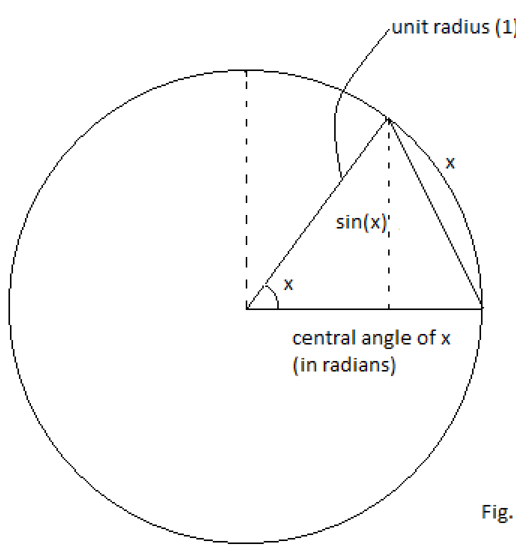

Fig. 1

We consider a unit circle, restricting our attention (for now) to angles x such that $0 \leq x < \frac{\pi}{2}$. As the figure illustrates, for a sector described by an angle in this range, the area of an inscribed triangle (whose height is $\sin(x)$, and whose base has unit length) is less than or equal to the area of the sector. But the area of the triangle must be given by $\frac{\sin(x)}{2}$, whereas (1) indicates that the area of the sector is given by $\frac{x}{2}$; and from this, it follows that

$$\sin(x) \leq x, \qquad 0 \leq x < \tfrac{\pi}{2}$$

From the trigonometric identity ($\sin(-x) =$ -$\sin(x)$), it follows similarly that for values of x such that $\frac{-\pi}{2} < x \leq 0$, $\sin(x)$ is larger than or equal to x (using the identity, $\sin(x) = -(-(\sin(x))) = -(\sin(-x))$; but for x in this range, we know already that $\sin(-x) \leq -x$; so $-\sin(-x) = \sin(x) \geq x$). In short, we find

$$0 \leq \sin(x) \leq x, \qquad 0 \leq x < \frac{\pi}{2} \tag{2}$$

$$x \leq \sin(x) \leq 0, \qquad -\frac{\pi}{2} < x \leq 0 \tag{3}$$

But from these the Squeeze Theorem tells us, since $\lim_{x\to 0} x = \lim_{x\to 0} 0 = 0$, that $\lim_{x\to 0}\sin(x) = 0$. (In detail, we must take each of the one-sided limits as x approaches zero; since these exist and are zero, the two-sided limit exists and is zero.)

From this we may the prove the fact, which we will shortly use, that $\lim_{x\to 0}\cos(x) = 1$.

Proof: Fix $\epsilon > 0$, and set $\delta_1 = \sqrt{2\epsilon}$. Our foregoing work indicates that

$$\exists\delta_2 : |\sin(t)| < \epsilon \text{ if } 0 < |t| < \delta_2$$

In particular, if we consider x satisfying $|x| < \delta_2$, then - since $\left|\frac{x}{2}\right| < |x|$ generally - we know that for any x on the interval, it is true that

$$\left|\sin(\tfrac{x}{2})\right| < \epsilon \text{ if } 0 < \left|\tfrac{x}{2}\right| < \delta_2$$

Therefore, for the δ in the limit statement

$$|\cos(x) - 1| < \epsilon \text{ if } 0 < |x| < \delta,$$

choose $\delta = \min(\delta_1, \delta_2)$. Then $|x| < \delta \Rightarrow |x|^2 < 2\epsilon$, and thus $\left|\frac{x}{2}\right|^2 < \frac{\epsilon}{2}$; and our selection of δ also implies that $\left|\sin^2(\frac{x}{2})\right| < \left|\frac{x}{2}\right|^2 < \frac{\epsilon}{2}$, so that $\left|2\sin^2(\frac{x}{2})\right| < \epsilon$. But the trigonometric identity $(1 - \cos(x) = 2\sin^2(\frac{x}{2}))$ indicates that we have

$$\left|2\sin^2(\tfrac{x}{2})\right| = |\cos(x) - 1| < \epsilon$$

which shows that our δ is satisfactory. ∎

With these limits in hand, we may additionally determine two more limits related to the trigonometric functions which will be key to later work. These are

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \tag{4}$$

$$\lim_{x\to 0} \frac{1 - \cos(x)}{x} = 0 \tag{5}$$

The proof of (4) has a geometric derivation which expands upon the derivation of $\lim_{x\to 0} \sin(x)$.

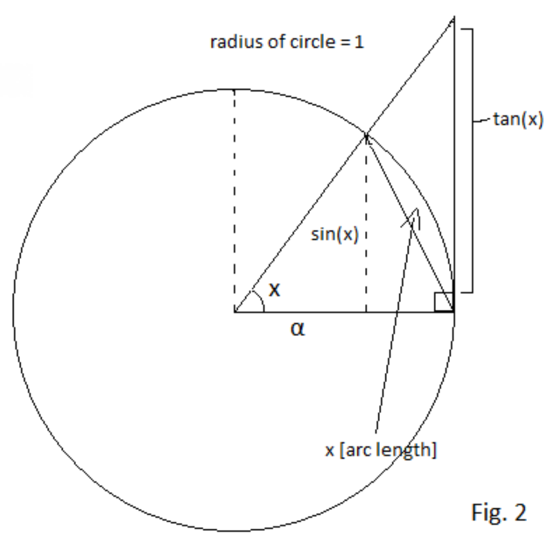

Fig. 2

Again inscribing a triangle in the unit circle, we extend this triangle so that the terminal point of its hypotenuse lies above that of the radial axis of the angle's measure, α. Thus the length of the opposite side in this triangle is $\tan(x)$. Note that the area of the sector falls in between $\frac{\sin(x)}{2}$ and $\frac{\tan(x)}{2}$ (these being the areas of the triangles inscribed in the circle, and engulfing it, respectively). Therefore, for x satisfying $0 < x < \frac{\pi}{2}$,

$$\frac{\sin(x)}{2} < \frac{x}{2} < \frac{\tan(x)}{2}$$

Multiplying by $\frac{2}{\sin(x)}$, we find

$$1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$$

and, on taking reciprocals, we arrive at

$$1 > \frac{\sin(x)}{x} > \cos(x)$$

To obtain a symmetric inequality for x satisfying $\frac{-\pi}{2} < x < 0$, note that $\tan(x)$ is an odd function; since we already have

$$\frac{\sin(-x)}{2} < \frac{-x}{2} < \frac{\tan(-x)}{2}$$

from our prior work, we take the negative of each constituent inequality, and find

$$\frac{\sin(x)}{2} > \frac{x}{2} > \frac{\tan(x)}{2}$$

Since $\sin(x) < 0$ in this context, the step of multiplying by $\frac{2}{\sin(x)}$ reverses the inequalities once more:

$$1 < \frac{x}{\sin(x)} < \frac{1}{\cos(x)}$$

and therefore taking reciprocals leaves us with the identical result. Thus,

$$1 > \frac{\sin(x)}{x} > \cos(x) \qquad \text{if} \qquad 0 < |x| < \tfrac{\pi}{2}$$

Now - to make explicit how the Squeeze Theorem may be applied here to derive (4) - we may fix $\epsilon > 0$. For this ϵ, since $\lim_{x\to 0} 1 = \lim_{x\to 0}\cos(x) = 1$, we know that there exist $\delta_1, \delta_2 > 0$ such that $|0| < \epsilon$ if $0 < |x| < \delta_1$ and $|\cos(x) - 1| < \epsilon$ if $0 < |x| < \delta_2$. Choosing $\delta = \min(\delta_1, \delta_2, \frac{\pi}{2})$, we find that if $0 < |x| < \delta$, then $1 > \dfrac{\sin(x)}{x} > \cos(x)$, and so $1 + \epsilon > 1 > \dfrac{\sin(x)}{x} > \cos(x) > 1 - \epsilon$.

Therefore $\left|\frac{\sin(x)}{x} - 1\right| < \epsilon$ if $0 < |x\ | < \delta$. This proves (4). (5) follows easily:

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos(x)}{x} &= \lim_{x\to 0}\frac{(1-\cos(x))(1+\cos(x))}{(x)(1+\cos(x))} = \lim_{x\to 0}\frac{\sin^2(x)}{(x)(1+\cos(x))}\\
&= (1)(\lim_{x\to 0}\frac{\sin(x)}{1+\cos(x)})\\
&= 0
\end{aligned}$$

We are now ready to prove the following theorems:

(1) *Theorem*: The six trigonometric functions are continuous on their natural domains.
(2) *Theorem*: The six trigonometric functions are differentiable on their natural domains.

Proof of (1): We show first that $\sin(x)$ is continuous everywhere.

We say that $\sin(x)$ is continuous at x_0 if

$$\lim_{x\to x_0}\sin(x) = \sin(x_0)$$

If we put $h = x - x_0$, then this is equivalent to

$$\lim_{h\to 0}\sin(x_0 + h) = \sin(x_0)$$

But, by the trigonometric identity for $\sin(\alpha + \beta)$, and by our previous work,

$$\begin{aligned}
\lim_{h \to 0} \sin(x_0 + h) &= \lim_{h \to 0} [\sin(x_0)\cos(h) + \sin(h)\cos(x_0)] \\
&= \lim_{h \to 0} \sin(x_0) \lim_{h \to 0} \cos(h) + \lim_{h \to 0} \sin(h) \lim_{h \to 0} \cos(x_0) \\
&= (\sin(x_0))(1) + (0)(\cos(x_0)) \\
&= \sin(x_0)
\end{aligned}$$

Thus $\sin(x)$ is continuous for any arbitrary value in its domain, since the identity and limit statements which we used hold true everywhere in its domain. Since the natural domain of $\sin(x)$ includes all real values (our definition of $\sin(x)$ indicates that it is a periodic function), $\sin(x)$ is therefore continuous everywhere.

Second, we show that $\cos(x)$ is continuous everywhere. As with $\sin(x)$, we may define the condition for continuity of $\cos(x)$ at x_0 by our variable h; i.e.,

$$\lim_{h \to 0} \cos(x_0 + h) = \cos(x_0)$$

implies continuity at this point. Using the trigonometric identity for $\cos(\alpha + \beta)$, and our previous work,

$$\begin{aligned}
\lim_{h \to 0} \cos(x_0 + h) &= \lim_{h \to 0} [\cos(x_0)\cos(h) - \sin(h)\sin(x_0)] \\
&= \lim_{h \to 0} \cos(x_0) \lim_{h \to 0} \cos(h) - \lim_{h \to 0} \sin(h) \lim_{h \to 0} \sin(x_0) \\
&= (\cos(x_0))(1) - (0)(\sin(x_0)) \\
&= \cos(x_0)
\end{aligned}$$

Given the identically periodic nature (in relation to $\sin(x)$) of $\cos(x)$, we conclude similarly that $\cos(x)$ is continuous everywhere. Finally, we note that, since the other trigonometric functions can be defined as ratios or reciprocals of $\sin(x)$ and $\cos(x)$, the continuity of $\sin(x)$ and $\cos(x)$ guarantees that these ratios are also continuous where defined; and the equivalence of these definitions with our original ones (based upon triangles inscribed in circles) implies that the natural domain of these functions is coincident with where these ratios are defined. Therefore, we conclude that these functions - and thus, all trigonometric functions - are continuous on their natural domains. ∎

Proof of (2): We show only that $\sin(x)$ and $\cos(x)$ are differentiable everywhere; it then follows unambiguously from the Quotient Rule for derivatives that the other trigonometric functions are differentiable on their natural domains. (Their formulaic derivatives may be determined by use of the Quotient Rule as well.)

For $\sin(x)$, the derivative exists and is given by

$$\begin{aligned}
\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} &= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)-\sin(x)}{h}+\lim_{h\to 0}\frac{\sin(h)\cos(x)}{h}\\
&= -\sin(x)\lim_{h\to 0}\frac{1-\cos(h)}{h}+\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}\\
&= \cos(x)
\end{aligned}$$

and for $\cos(x)$,

$$\begin{aligned}
\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h} &= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\sin(h)\sin(x)-\cos(x)}{h}\\
&= \lim_{h\to 0}\frac{\cos(x)\cos(h)-\cos(x)}{h}-\lim_{h\to 0}\frac{\sin(h)\sin(x)}{h}\\
&= -\cos(x)\lim_{h\to 0}\frac{1-\cos(h)}{h}-\sin(x)\lim_{h\to 0}\frac{\sin(h)}{h}\\
&= -\sin(x)
\end{aligned}$$

and therefore both $\sin(x)$ and $\cos(x)$ are differentiable. ∎