> The acceleration induced in one body by another is some definite function of their positions, velocities and internal structure, and is unaffected by the presence of other bodies. In a many-body system, the acceleration of any given body is equal to the sum of the accelerations induced in it by each of the other bodies individually.

These laws, which appear in a rather unfamiliar form, are actually completely equivalent to Newton's laws, as stated in the previous section. In view of the apparently fundamental role played by the concept of force in Newtonian mechanics, it is remarkable that we have been able to reformulate the basic laws without mentioning this concept. It can of course be introduced, by defining it through Newton's second law, (1.1). The utility of this definition arises from the fact that forces satisfy Newton's third law, (1.3), while accelerations satisfy only the more complicated law, (1.7). Since the mutually induced accelerations of two given bodies are always proportional, they are essentially determined by a single function, and it is useful to introduce the more symmetric concept of force, for which this becomes obvious.

It is interesting to note, finally, that one consequence of our basic laws is the additive nature of mass. Let us take a three-body system. Then, returning to the notation of the previous section, the equations of motion for the three bodies are

$$\begin{aligned} m_1\boldsymbol{a}_1 &= \boldsymbol{F}_{12} + \boldsymbol{F}_{13}, \\ m_2\boldsymbol{a}_2 &= \boldsymbol{F}_{21} + \boldsymbol{F}_{23}, \\ m_3\boldsymbol{a}_3 &= \boldsymbol{F}_{31} + \boldsymbol{F}_{32}. \end{aligned} \tag{1.9}$$

If we add these equations, then, in view of (1.3), the terms on the right cancel in pairs, and we are left with

$$m_1\boldsymbol{a}_1 + m_2\boldsymbol{a}_2 + m_3\boldsymbol{a}_3 = \boldsymbol{0}, \tag{1.10}$$

which is the generalization of (1.7). Now, if we suppose that the force between the second and third is such that they are rigidly bound together to form a composite body, their accelerations must be equal: $\boldsymbol{a}_2 = \boldsymbol{a}_3$. In that case, we get

$$m_1\boldsymbol{a}_1 = -(m_2 + m_3)\boldsymbol{a}_2,$$

which shows that the mass of the composite body is just $m_{23} = m_2 + m_3$.

1.4 External Forces

To find the motion of the various bodies in any dynamical system, we have to solve two closely interrelated problems. First, given the positions and velocities at any one instant of time, we have to determine the forces acting on each body. Second, given the forces acting, we have to compute the new positions and velocities after a short interval of time has elapsed. In a general case, these two problems are inextricably bound up with each other, and must be solved simultaneously. If, however, we are concerned with the motions of a small body, or group of small bodies, then we can often neglect its effect on other bodies, and in that case the two problems can be separated.

For example, in discussing the motion of an artificial satellite, we can clearly ignore its effect on the Earth. Since the motion of the Earth is already known, we can calculate the force on the satellite as a function of its position and (if atmospheric resistance is included) its velocity. Then, taking the force as known, we can solve separately the problem of its motion. In the latter problem, we are really concerned with the satellite alone. The Earth enters simply as a known external influence.

In many cases, therefore, it is useful to concentrate our attention on a small part of a dynamical system, and to represent the effect of everything outside this by external forces, which we suppose to be known in advance, as functions of position, velocity and time. This is the kind of problem with which we shall be mainly concerned in the next few chapters. Typically, we shall consider the motion of a particle under a known external force. In Chapter 6, we consider, for the gravitational and electrostatic cases, the complementary problem of determining the force from a knowledge of the positions of other bodies. Later, in Chapter 7, we return to the more complex type of problem in which the system of immediate interest cannot be taken to be merely a single particle.

1.5 Summary

To some extent, the selection of a group of basic concepts, in terms of which others are to be defined, is a matter of choice. We have chosen to regard position and time (relative to some frame of reference) as basic. From this point of view, Newton's laws must be regarded as containing definitions in addition to physical laws. The first law contains the definition of an inertial and the principle of relativity discussed in §1.1. This arises from the fact that if the speed of light is constant with respect to one inertial frame — as it should be according to electromagnetic theory — then the usual rules for combining velocities would lead to the conclusion that it is not constant with respect to a relatively moving frame, in contradiction with the statement that all inertial frames are equivalent. This paradox can only be resolved by the introduction of Einstein's theory of relativity (*i.e.*, 'special' relativity). Classical electromagnetic theory and classical mechanics *can* be incorporated into a single self-consistent theory, but only by ignoring the relativity principle and sticking to one 'preferred' inertial frame.

1.3 The Concepts of Mass and Force

It is an important general principle of physics (though not universally applied!) that no quantity should be introduced into the theory which cannot, at least in principle, be measured. Now, Newton's laws involve not only the concepts of velocity and acceleration, which can be measured by measuring distances and times, but also the new concepts of mass and force. To give the laws a physical meaning we have, therefore, to show that these are measurable quantities. This is not quite as trivial as it might seem, because any experiment designed to measure these quantities must necessarily involve Newton's laws themselves in its interpretation. Thus the operational definitions of mass and force — the prescriptions of how they may be measured — which are required to make the laws physically significant, are actually contained in the laws themselves. This is by no means an unusual or logically objectionable situation, but it may clarify the status of these concepts to reformulate the laws in such a way as to isolate their definitional element.

Let us consider first the measurement of mass. Since the units of mass are arbitrary, we have to specify a way of comparing the masses of two given bodies. It is important to realize that we are discussing here the *inertial* mass, which appears in Newton's second law, (1.1) and not the *gravitational* mass, which appears in (1.5). The two are of course proportional, but this *equivalence principle* is a physical law derived from experimental observation (in particular from Galileo's observations of falling bodies, from which he deduced that in a vacuum all bodies would fall equally fast) rather than an *a priori* assumption. To verify the law, we must be able to measure each kind of mass separately. This rules out, for example, the use of a balance, which compares gravitational masses.

Clearly, we can compare the inertial masses of two bodies by subjecting them to equal forces and comparing their accelerations, but this does not help unless we have some way of knowing that the forces *are* equal. However there is one case in which we *do* know this, because of Newton's third law. If we isolate the two bodies from all other matter, and compare their mutually induced accelerations, then according to (1.1) and (1.3),

$$m_1 \boldsymbol{a}_1 = -m_2 \boldsymbol{a}_2, \tag{1.7}$$

so that the accelerations are oppositely directed, and inversely proportional to the masses. If we allow two small bodies to collide, then during the collision the effects of more remote bodies are generally negligible in comparison with their effect on each other, and we may treat them approximately as an isolated system. (Such collisions will be discussed in detail in Chapters 2 and 7.) The mass ratio can then be determined from measurements of their velocities before and after the collision, by using (1.7) or its immediate consequence, the law of *conservation of momentum*,

$$m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2 = \text{constant}. \tag{1.8}$$

If we wish to separate the definition of mass from the physical content of equation (1.7), we may adopt as a fundamental axiom the following:

> In an isolated two-body system, the accelerations always satisfy the relation $\boldsymbol{a}_1 = -k_{21}\boldsymbol{a}_2$, where the scalar k_{21} is, for two given bodies, a constant independent of their positions, velocities and internal states.

If we choose the first body to be a standard body, and conventionally assign it unit mass (say $m_1 = 1$ kg), then we may *define* the mass of the second to be k_{21} in units of this standard mass (here $m_2 = k_{21}$ kg).

Note that for consistency, we must have $k_{12} = 1/k_{21}$. We must also assume of course that if we compare the masses of three bodies in this way, we obtain consistent results:

> For any three bodies, the constants k_{ij} satisfy $k_{31} = k_{32}k_{21}$.

It then follows that for *any* two bodies, k_{32} is the mass ratio: $k_{32} = m_3/m_2$.

To complete the list of fundamental axioms, we need one which deals with systems containing more than two bodies, analogous to the law of composition of forces, (1.2). This may be stated as follows:

that if we are interested in the overall motion of even a very large object, such as a planet, we may often legitimately treat it as a point particle located at the *centre of mass* of the body. The laws themselves prescribe the meaning of the 'position' of an extended body.

We shall begin by simply stating Newton's laws, and defer to the following section a discussion of the physical significance of the concepts involved, particularly those of *mass* and *force*.

Let us consider an isolated system comprising N bodies, which we label by an index $i = 1, 2, \ldots, N$. By saying that the system is *isolated*, we mean that all other bodies are sufficiently remote to have a negligible influence on it. Each of the N bodies is assumed to be small enough to be treated as a point particle. The position of the ith body with respect to a given inertial frame will be denoted by $\boldsymbol{r}_i(t)$. Its velocity and acceleration are

$$\begin{aligned} \boldsymbol{v}_i(t) &= \dot{\boldsymbol{r}}_i(t), \\ \boldsymbol{a}_i(t) &= \dot{\boldsymbol{v}}_i(t) = \ddot{\boldsymbol{r}}_i(t), \end{aligned}$$

where the dots denote differentiation with respect to the time t. For example

$$\dot{\boldsymbol{r}} \equiv \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}.$$

Each body is characterized by a scalar constant, its *mass* m_i. Its momentum $\boldsymbol{p}_i$ is defined to be mass × velocity:

$$\boldsymbol{p}_i = m_i \boldsymbol{v}_i.$$

The equation of motion, which specifies how the body will move is *Newton's second law* (mass × acceleration = force):

$$\dot{\boldsymbol{p}}_i = m_i \boldsymbol{a}_i = \boldsymbol{F}_i, \tag{1.1}$$

where $\boldsymbol{F}_i$ is the total force acting on the body. This force is composed of a sum of forces due to each of the other bodies in the system. If we denote the force *on* the ith body *due to* the jth body by $\boldsymbol{F}_{ij}$, then

$$\boldsymbol{F}_i = \boldsymbol{F}_{i1} + \boldsymbol{F}_{i2} + \cdots + \boldsymbol{F}_{iN} = \sum_{j=1}^{N} \boldsymbol{F}_{ij}, \tag{1.2}$$

where of course $\boldsymbol{F}_{ii} = \boldsymbol{0}$, since there is no force on the ith body due to itself. Note that since the sum on the right side of (1.2) is a vector sum, this equation incorporates the 'parallelogram law' of composition of forces.

The two-body forces $\boldsymbol{F}_{ij}$ must satisfy *Newton's third law*, which asserts that 'action' and 'reaction' are equal and opposite,

$$\boldsymbol{F}_{ji} = -\boldsymbol{F}_{ij}. \tag{1.3}$$

Moreover, $\boldsymbol{F}_{ij}$ is a function of the positions and velocities (and internal structure) of the ith and jth bodies, but is unaffected by the presence of the other bodies. (It can be argued that this is an unnecessarily restrictive assumption. It would be perfectly possible to include also, say, three-body forces, which depend on the positions and velocities of three particles simultaneously. However, within the realm of validity of classical mechanics, no such forces are known, and their inclusion would be an inessential complication.) Because of the relativity principle, the force can in fact depend only on the *relative* position

$$\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$$

(see Fig. 1.1), and the *relative* velocity

$$\boldsymbol{v}_{ij} = \boldsymbol{v}_i - \boldsymbol{v}_j.$$

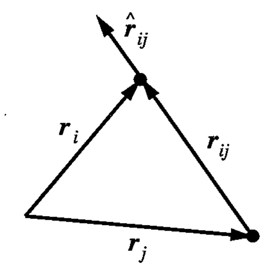

Fig. 1.1

If the forces are known, as functions of the positions and velocities, then from (1.1) we can predict the future motion of the bodies. Given their initial positions and velocities, we can solve these equations (analytically or numerically) to find their positions at a later time.

There is here an implicit assumption of perfect knowledge and infinite precision of calculation. It is now recognized (see Chapters 13, 14) that this assumption is, in general, false, leading to a loss of predictability. However, for the time being, we shall assume that our solution can be effected.