Physics Forum Problem in Linear Independence

Given

$$A = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \tag{1}$$

in $\mathbb{R}^n$ is linearly independent, prove that

$$B = \begin{bmatrix} \mathbf{u}-\mathbf{v} & \mathbf{u}+\mathbf{w} & \mathbf{v}+\mathbf{w} \end{bmatrix} \tag{2}$$

is also linearly independent. I gather it is. Suppose there exist scalars b_1, b_2, b_3 such that

$$b_1\left[\mathbf{u}-\mathbf{v}\right] + b_2\left[\mathbf{u}+\mathbf{w}\right] + b_3\left[\mathbf{v}+\mathbf{w}\right] = \mathbf{0} \tag{3}$$

$$\begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix} \begin{bmatrix} b_1 + b_2 \\ -b_1 + b_3 \\ b_2 + b_3 \end{bmatrix} = \mathbf{0} \tag{4}$$

Since A is linearly indepenent, this equation is true if and only if

$$b_1 + b_2 = 0 \tag{5}$$

$$-b_1 + b_3 = 0 \tag{6}$$

$$b_2 + b_3 = 0 \tag{7}$$

As a matrix, this is

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mathbf{0} \tag{8}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

$$\left[\mathbf{u}-\mathbf{v}\right] - \left[\mathbf{u}+\mathbf{w}\right] + \left[\mathbf{v}+\mathbf{w}\right] = \mathbf{0} \tag{10}$$

That looks like a linear dependence relationship to me,

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{11}$$