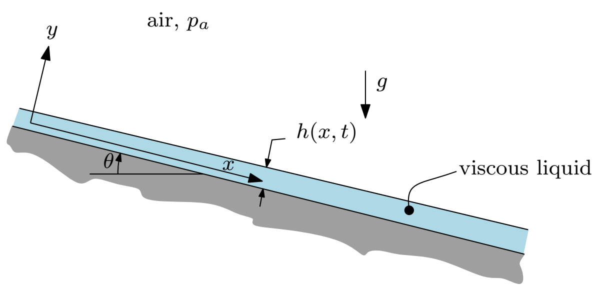
Problem 6.16

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



A rigid plane surface is inclined at an angle θ relative to the horizontal and wetted by a thin layer of highly viscous liquid which begins to flow down the incline.

(a) Show that if the flow is two-dimensional and in the inertia-free limit, and if the angle of the inclination is not too small, the local thickness $h(x,t)$ of the liquid layer obeys the equation

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = 0$$

where

$$c = \frac{\rho g h^2}{\mu}\sin\theta$$

(b) Demonstrate that the result of (a) implies that in a region where h decreases in the flow direction, the angle of the free surface relative to the inclined plane will steepen as the fluid flows down the incline, while in a region where h increases in the flow direction, the reverse is true. Does this explain something about what happens to slow-drying paint when it is applied to an inclined surface?

(c) Considering the result of (b) above, do you think that the steady-state solutions of the previous problems would ever apply in practice? Discuss.

Solution:

(a) Assumptions:

- $\mathrm{Re}_H \frac{H}{L} \ll 1$
- two dimensional flow
- THIN layer $\Rightarrow \frac{H}{L} = \frac{\text{characteristic height}}{\text{characteristic length}} \ll 1$

Unknown: $h(x,t)$?

For the sake of completeness, this solution provides detailed nondimensionalization of full 2D Navier-Stokes as well as continuity equations. So please bear with me!!

(1) Choose relevant scales: (* denotes dimensionless variables)

$$x^* = \frac{x}{L} \qquad v_x^* = \frac{v_x}{U} \qquad t^* = \frac{tU}{L}$$

$$y^* = \frac{y}{H} \qquad v_y^* = \frac{v_y}{V} \qquad p^* = \frac{p}{\mathcal{P}}$$

where $\mathcal{P}$ is an unknown pressure scale.

(2) Non-dimensionalize continuity:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0$$
$$\frac{U}{L}\frac{\partial v_x^*}{\partial x^*} + \frac{V}{L}\frac{\partial v_y^*}{\partial y^*} = 0$$

Since dimensionless variables are assumed to be of the same order ($\mathcal{O}(1)$),

$$\frac{U}{L} \sim \frac{V}{H}$$
$$\Rightarrow V = \frac{H}{L}U \qquad \text{where} \quad \frac{H}{L} \ll 1$$

(3) Non-dimensionalize Navier-Stokes:

<u>x-direction:</u>

$$\rho\frac{U^2}{L}\left(\frac{\partial v_x^*}{\partial t^*} + v_x^*\frac{\partial v_x^*}{\partial x^*} + v_y^*\frac{\partial v_x^*}{\partial y^*}\right) = -\frac{\mathcal{P}}{L}\frac{\partial p^*}{\partial x^*} + \rho g\sin\theta + \mu\left(\frac{U}{L^2}\frac{\partial^2 v_x^*}{\partial x^{*2}} + \frac{U}{H^2}\frac{\partial^2 v_x^*}{\partial y^{*2}}\right)$$

Divide through by $\frac{\mu U}{H^2}$:

$$\underbrace{\frac{\rho U H^2}{\mu L}}_{\mathrm{Re}_H\frac{H}{L}=\text{small}}\cancel{\left(\frac{\partial v_x^*}{\partial t^*} + v_x^*\frac{\partial v_x^*}{\partial x^*} + v_y^*\frac{\partial v_x^*}{\partial y^*}\right)} = -\frac{\mathcal{P}H^2}{\mu U L}\frac{\partial p^*}{\partial x^*} + \frac{\rho g H^2\sin\theta}{\mu U} + \underbrace{\left(\frac{H}{L}\right)^2}_{\text{small}}\cancel{\frac{\partial^2 v_x^*}{\partial x^{*2}}} + \frac{\partial^2 v_x^*}{\partial y^{*2}}$$

<u>y-direction:</u>

$$\rho\frac{U^2}{L}\frac{H}{L}\left(\frac{\partial v_y^*}{\partial t^*} + v_x^*\frac{\partial v_y^*}{\partial x^*} + v_y^*\frac{\partial v_y^*}{\partial y^*}\right) = -\frac{\mathcal{P}}{H}\frac{\partial p^*}{\partial y^*} - \rho g\cos\theta + \mu\frac{H}{L}\left(\frac{U}{L^2}\frac{\partial^2 v_y^*}{\partial x^{*2}} + \frac{U}{H^2}\frac{\partial^2 v_y^*}{\partial y^{*2}}\right)$$

Divide through by $\frac{\mu U}{H^2}$:

$$\underbrace{\frac{\rho U H^2}{\mu L}}_{\mathrm{Re}_H\left(\frac{H}{L}\right)^2=\text{small}}\cancel{\left(\frac{\partial v_y^*}{\partial t^*} + v_x^*\frac{\partial v_y^*}{\partial x^*} + v_y^*\frac{\partial v_y^*}{\partial y^*}\right)} = -\frac{\mathcal{P}H}{\mu U}\frac{\partial p^*}{\partial y^*} - \frac{\rho g H^2\cos\theta}{\mu U} + \underbrace{\left(\frac{H}{L}\right)^3}_{\text{small}}\cancel{\frac{\partial^2 v_y^*}{\partial x^{*2}}} + \underbrace{\frac{H}{L}}_{\text{small}}\cancel{\frac{\partial^2 v_y^*}{\partial y^{*2}}}$$

Now assume that $\frac{PH}{\mu U} \sim \mathcal{O}(1)$:[1]

$$\mathcal{P} = \frac{\mu U}{H} \Leftarrow \text{viscous pressure scale for low Re}$$

Substitute this expression for $\mathcal{P}$ in x-direction equation:

$$0 = -\frac{\mu U}{H}\frac{H^2}{\mu U L}\frac{\partial p^*}{\partial x^*} + \frac{\rho g H^2\sin\theta}{\mu U} + \frac{\partial^2 v_x^*}{\partial y^{*2}}$$
$$0 = -\underbrace{\frac{H}{L}}_{\text{small}}\cancel{\frac{\partial p^*}{\partial x^*}} + \frac{\rho g H^2\sin\theta}{\mu U} + \frac{\partial^2 v_x^*}{\partial y^{*2}}$$

$$\Rightarrow \underbrace{\boxed{0 = \frac{\rho g H^2\sin\theta}{\mu U} + \frac{\partial^2 v_x^*}{\partial y^{*2}}}}_{x-\text{momentum}} \qquad \text{and} \qquad \underbrace{\boxed{0 = \frac{\partial p^*}{\partial y^*} + \frac{\rho g H^2\cos\theta}{\mu U}}}_{y-\text{momentum}}$$

Going back to the dimensional form,

$$0 = \rho g\sin\theta + \mu\frac{\partial^2 v_x}{\partial y^2} \tag{6.16a}$$

[1] Notice we now only have a hydrostatic relation in y-direction

(4) Solve for v_x by integrating both sides of Eq. (6.16a):

$$\int \frac{\partial^2 v_x}{\partial y^2} dy = -\int \frac{\rho g \sin\theta}{\mu} dy$$
$$\frac{\partial^2 v_x}{\partial y^2} = -\frac{\rho g \sin\theta}{\mu} y + C_1$$

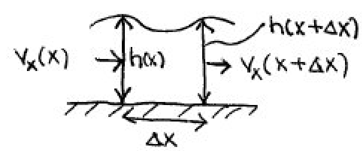
Using B.C. that $\frac{\partial v_x}{\partial y} = 0$ at $y = h$ (free surface),

$$C_1 = \frac{\rho g h \sin\theta}{\mu}$$
$$\Rightarrow \frac{\partial^2 v_x}{\partial y^2} = -\frac{\rho g \sin\theta}{\mu}(y - h)$$

Integrating again and using no-slip B.C. ($v_x = 0$ at $y = 0$):

$$v_x = \frac{\rho g \sin\theta}{\mu}(hy - \frac{1}{2}y^2) \tag{6.16b}$$

(5) Use mass conservation to obtain a single evolution equation for $h(x, t)$.
Consider the following control volume in the limit of $\Delta x \to 0$:



$$\frac{d}{dt}\int_{CV} \rho\, dV + \rho \int_{CS} (\mathbf{v} - \cancel{\mathbf{v}_c}) \cdot \hat{\mathbf{n}} dA = 0$$
$$\lim_{\Delta x \to 0} \frac{\cancel{\rho} \frac{d}{dt}\cancel{(h\Delta x)} + \cancel{\rho} \int_0^h v_x dy}{\cancel{\Delta x}} = 0$$
$$\Rightarrow \boxed{\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(\int_0^h v_x dy\right) = \frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0} \tag{6.16c}$$

The above equation can also be derived by combining the kinematic boundary condition, $\frac{\partial h}{\partial t} + v_x \frac{\partial h}{\partial x} = v_y|_{y=h(x)}$, with conservation of mass.

(6) Combine Eq. (6.16b) with Eq. (6.16c):

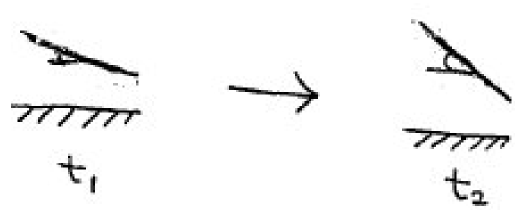
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h \frac{\rho g \sin\theta}{\mu}(hy - \frac{1}{2}y^2) dy = 0$$
$$\frac{\partial h}{\partial t} + \frac{\rho g \sin\theta}{\mu}\frac{\partial}{\partial x}\left(\frac{h^3}{3}\right) = 0$$
$$\boxed{\frac{\partial h}{\partial t} + \underbrace{\left(\frac{\rho g \sin\theta}{\mu} h^2\right)}_{c} \frac{\partial h}{\partial x} = 0} \tag{6.16d}$$

Eq. (6.16d) is a nonlinear wave equation with a solution of the form $h = f(x - ct)$, where c is the wave speed.

(b) Since $\frac{\rho g \sin\theta}{\mu} h^2 \geq 0$, $\frac{\partial h}{\partial t}$ and $\frac{\partial h}{\partial x}$ have opposite signs to satisfy Eq. (6.16d).

Thus, where h is decreasing locally ($\frac{\partial h}{\partial x} < 0$), h increases in time ($\frac{\partial h}{\partial t} > 0$).

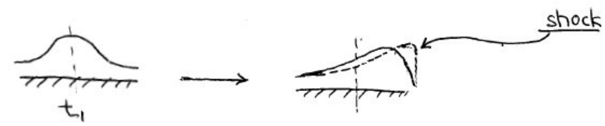
Angle of free surface steepens because points of larger h increase more rapidly ($c \sim h^2$) than points of lower h:



Where h is increasing locally ($\frac{\partial h}{\partial x} > 0$), h decreases in time ($\frac{\partial h}{\partial t} < 0$),

Angle of free surface flattens because points of larger h decrease more rapidly ($c \sim h^2$) than points of lower h:

$\therefore$ In the case of slow-drying paint, when there is a bump, Eq. (6.16d) dictates that the bump grows! However it never forms a shock because in reality, one has to consider effects of surface tension.



In practice, the solution to Eq. (6.16d) fails (or goes unstable) in the case of a symmetric perturbation, as explained in (b). Thus, it is not very applicable unless one accounts or effects of surface tension and such.

However, when h is monotonically increasing ($\frac{\partial h}{\partial x} > 0$ everywhere) the solution to Eq. (6.16d) is indeed stable since it predicts that h flattens in time.

$\square$

Problem Solution by Sungyon Lee, Fall 2005

Problem 6.21

(a) In terms of flow geometry, this problem is similar to 6.3; the difference here being that the flow is 'unsteady'!

Writing the Navier-Stokes equation in the cylindrical gap between the bearing pad and ground:

$$\rho\left(\underbrace{\frac{\partial u_r}{\partial t}}_{I}+\underbrace{u_r\frac{\partial u_r}{\partial r}}_{II}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}+\underbrace{u_z\frac{\partial u_r}{\partial z}}_{III}-\frac{u_\theta^{\,2}}{r}\right)=-\frac{\partial p}{\partial r}+\mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)}_{IV}+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}+\underbrace{\frac{\partial^2 u_r}{\partial z^2}}_{V}-\underbrace{\frac{u_r}{r^2}}_{VI}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right]$$

Let's perform an order-of-magnitude analysis. Note that $\frac{\partial}{\partial \theta}=0$.

$$\frac{IV}{V}\sim\frac{h^2}{D^2}<<1$$

(Note that terms *IV* and *VI* are of the same order, so that both vanish compared to *V* !)

$$\frac{II}{V}\sim\frac{\rho v_r D}{\mu}\left(\frac{h^2}{D^2}\right)<<1$$

Invoking the continuity equation in cylindrical coordinates, we have:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)+\frac{\partial u_z}{\partial z}=0\Rightarrow\frac{u_r}{r}\sim\frac{u_z}{z} \qquad (1)$$

Also, we have

$$\frac{II}{III}\sim\frac{\frac{u_r}{r}}{\frac{u_z}{z}} \qquad (2)$$

From (1) and (2), we can say that *II* and *III* are of the same order. Hence, both vanish compared to *V*.

Now, lets look at where *I* stands! We have:

$$\frac{I}{V}\sim\frac{\rho h^2}{\mu\tau}$$

where, τ is the time scale involved in the this process. The source of unsteadiness is the pad settling down, which renders u_r and other flow variables time-dependent. Hence,

$$\tau\sim\frac{h}{S}\Rightarrow\frac{I}{V}\sim\frac{\rho S h}{\mu}$$

Since it is given that *S* is very small and also *h* is small, we can safely assume that *I* can be neglected compared to *V*. Hence we have the N-S eqn as:

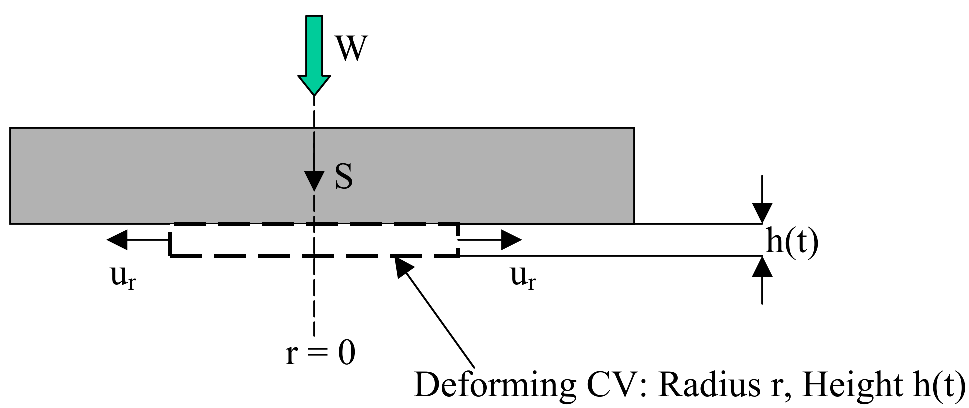
$$-\frac{\partial p}{\partial r}+\mu\frac{\partial^2 u_r}{\partial z^2}=0$$

Integrating, we get $u_r(z)$ as:

$$u_r=\frac{1}{2\mu}\left(-\frac{\partial p}{\partial r}\right)\left(zh-z^2\right)$$

$$\Rightarrow Q(r)=2\pi r\int_0^h u_r(z)dz=\frac{h^3\pi r}{6\mu}\left(-\frac{\partial p}{\partial r}\right)=\frac{h^3\pi r}{6\mu}\left(-\frac{dp}{dr}\right) \tag{3}$$

Note that *Q* is a function of *r*, since the settling down of the pad drives greater and greater flow rates as *r* increases! This can be verified by applying mass conservation in a cylindrical *CV*, as shown. The height of this *CV* changes as *h(t)*.



$$0=\frac{d}{dt}\int_{CV}\rho dV+\int_{CS}\rho(\vec{V}-\vec{V}_{CS}).\vec{n}dA$$

$$\Rightarrow\frac{d}{dt}\left(\pi r^2h\right)+2\pi rhu_r=0\Rightarrow-\pi r^2\frac{dh}{dt}=2\pi rhu_r=Q(r)$$

Since, $\frac{dh}{dt}=-S$, we have:

$Q(r)=\pi r^2S$

Plugging Q(r) into (3), we have:

$$\pi r^2S=\frac{h^3\pi r}{6\mu}\left(-\frac{dp}{dr}\right)\Rightarrow dp=-\frac{6\mu Sr}{h^3}dr$$

$$\Rightarrow p(r) = \frac{3\mu S}{h^3}\left(\frac{D^2}{4} - r^2\right)$$

where the BC used is $p\left(\frac{D}{2}\right) = 0$ (gauge pressure!)

Now, we can perform a vertical force-balance on the pad:

$$W = \frac{3\mu S}{h^3}\int_0^{\frac{D}{2}}\left(\frac{D^2}{4} - r^2\right)2\pi r dr = \frac{3\pi\mu S D^4}{32h^3}$$

$$\Rightarrow \boxed{S = \frac{32Wh^3}{3\pi\mu D^4}} \tag{4}$$

(b) Plugging in the numbers, we obtain

$$S = \frac{32\times 100\times 27\times 10^{-15}}{3\pi\times 0.93\times 81\times 10^{-4}} = 1.2\times 10^{-9}\, m/s \quad \text{(very small!)}$$

(c) From (4), we have:

$$S = -\frac{dh}{dt} = \frac{32Wh^3}{3\pi\mu D^4} \Rightarrow -\int_{h_0}^{h}\frac{dh}{h^3} = \int_0^t \frac{32W}{3\pi\mu D^4}dt \tag{5}$$

$$\Rightarrow \frac{1}{h^2} - \frac{1}{h_0^{\,2}} = \frac{64Wt}{3\pi\mu D^4} \Rightarrow \boxed{\frac{h}{h_0} = \left[1 + \frac{64Wh_0^{\,2}}{3\pi\mu D^4}t\right]^{-1/2}}$$

(d) $2h = h_0$. Plugging in the values, we get the time required as 10.4 hours.

(e) Referring to relation (5), $S = \frac{dh}{dt}$ instead of $-\frac{dh}{dt}$ and consequently, we have:

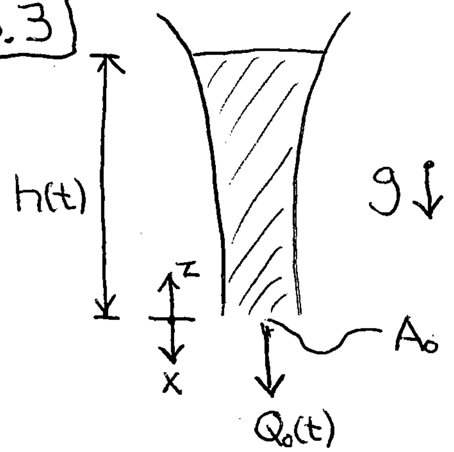
$$\frac{1}{h_0^{\,2}} - \frac{1}{h^2} = \frac{64Ft}{3\pi\mu D^4}$$

As the disk is pulled away, $h \to \infty$ and we have t_∞ as:

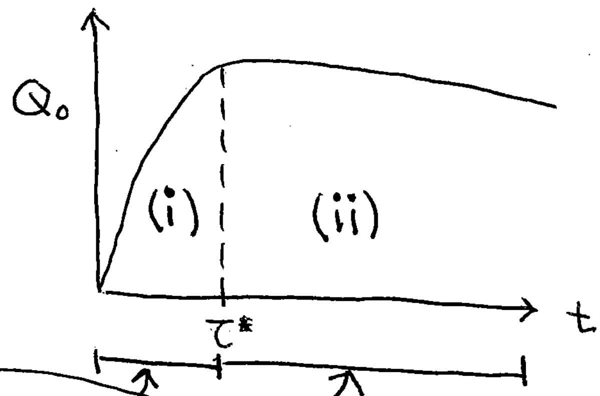
$$\boxed{t_\infty = \frac{3\pi\mu D^4}{64h_0^{\,2}F}}$$

Solution by Mayank Kumar, Fall 2007

8.3



ⓐ



(i) Before t^*, Q_0 increases to a maximum value since flow has to accelerate from 0 to a certain value. In this region, the <u>inertia</u> terms (namely, $\rho \frac{\partial v_x}{\partial t}$) dominate, thus explaining a roughly linear increase in Q_0 with time.

(ii) Once Q_0 reaches its maximum value, the viscous effects become more important ($\frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right)$); therefore, Q_0 decreases slowly.

ⓑ At $t = t^*$, we may assume that $\rho \frac{\partial v_x}{\partial t} \sim \frac{\mu}{r} \cdot \frac{\partial}{\partial r}\left(r \frac{\partial v_x}{\partial r}\right)$ based on our reasoning in ⓐ.

$$\therefore \quad \rho \frac{U}{\tau^*} \cdot \underbrace{\frac{\partial v_x^*}{\partial t^*}} \sim \frac{\mu U}{R_0^2} \cdot \underbrace{\frac{1}{r^*}\frac{\partial}{\partial r^*}\left(r^* \frac{\partial v_x^*}{\partial r^*}\right)}$$

dimensionless $\Rightarrow \sim O(1)$

$$\therefore \quad \rho \frac{U}{t^*} \sim \frac{\mu U}{R_0^2} \quad \text{OR} \quad \rho \frac{\cancel{U}}{t^*} \sim \frac{\mu \cancel{U}}{A_0}$$

$$\therefore \quad \boxed{\tau^* \sim \frac{A_0}{\nu}}$$

← This is equivalent to <u>diffusion time scale</u> that measures how long it takes for fluid particles in the middle of the pipe to feel wall effects!

(c)

inertia-free $\Rightarrow$ neglect $\frac{D\underline{v}}{Dt}$ terms

gradually varying A $\Rightarrow$ Assume locally fully-developed

$\Rightarrow$ N.S. equations reduce to

$$\boxed{\begin{aligned} 0 &= -\frac{\partial P}{\partial z} + \frac{\mu}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) - \rho g \\ 0 &= \frac{\partial P}{\partial r} \quad \Rightarrow P \neq P(r) \end{aligned}} \qquad (8.3c-1)$$

(* Please refer back to Solution 6.16 for details of systematically reducing Navier-Stokes equations.)

- Use separation of variables on (8.3c-1) to solve for V_z:

$$\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right) = \frac{r}{\mu}\left(\rho g + \frac{\partial P}{\partial z}\right)$$

$$\frac{\partial V_z}{\partial r} = \frac{r}{2\mu}\left(\rho g + \frac{\partial P}{\partial z}\right) + \frac{C_1}{r} \qquad V_z \equiv \text{finite at } r=0$$

$$V_z = \frac{r^2}{4\mu}\left(\rho g + \frac{\partial P}{\partial z}\right) + \cancel{C_1 \ln r} + C_2$$

$$V_z(r=R) = 0 = \frac{R^2(z)}{4\mu}\left(\rho g + \frac{\partial P}{\partial z}\right) + C_2$$

$$\therefore \boxed{V_z = \frac{1}{4\mu}\left(\rho g + \frac{\partial P}{\partial z}\right)\left(r^2 - R^2(z)\right)} \qquad (8.3c-2)$$

- Look at mass conservation :

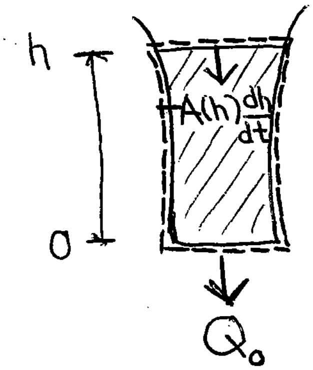
$$Q(t) = Q_0(t) = \int_0^{R(z)} (-) V_z \, 2\pi r \, dr$$

due to the way $\hat{e}_z$ and the direction of flow are defined

$$= -\frac{\pi}{2\mu}\int_0^{R(z)} \left(\rho g + \frac{\partial P}{\partial z}\right)\left(r^3 - R^2(z) r\right) dr$$

$$= \frac{\pi R^4(z)}{8\mu}\left(\rho g + \frac{\partial P}{\partial z}\right)$$

Consider the following control volume.



By mass conservation,

$$\underbrace{A(h)\frac{dh}{dt}}_{\approx \text{ the rate of CV increasing}} + \underbrace{Q_o}_{\text{the rate of mass exiting CV}} = 0$$

$$\Rightarrow Q_o = -A(h)\frac{dh}{dt} = \frac{\pi R^4(z)}{8\mu}\left(\rho g + \frac{\partial P}{\partial z}\right)$$

OR

Since $A(z) = \pi R^2(z)$,

$$-A(h)\frac{dh}{dt} = \frac{A^2(z)}{8\pi\mu}\left(\rho g + \frac{\partial P}{\partial z}\right)(= Q_o) \qquad (8.3C-3)$$

- Solve for pressure by integrating wrt z on both sides:

$$\int_{P(z=0)}^{P(z=h)} dP = -\int_0^{h(t)} \rho g\, dz - 8\pi\mu A(h)\frac{dh}{dt}\int_0^{h(t)} \frac{dz}{A^2(z)}$$

$$\cancel{P_{atm} \to P_{atm}} = -\rho g h(t) - 8\pi\mu A(h)\frac{dh}{dt}\int_0^{h(t)} \frac{dz}{A^2(z)}$$

By re-arranging,

$$\boxed{\frac{dh}{dt} = -\frac{g h(t)}{8\pi\nu A(h)}\left[\int_0^{h(t)} \frac{dz}{A^2(z)}\right]^{-1}} \qquad (8.3C-4)$$

(d) With solution in (c) given, check if intertia effects are indeed negligible in comparison to viscous terms:

(In other words, $\frac{A}{\nu\tau} \ll 1$??)

$*$ Keep in mind that H is the big length scale and $\sqrt{A}$ the small!!
$\Rightarrow A/H^2 \ll 1$

- Obtain the characteristic time scale, τ, by non-dimensionalizing (8.3c-4);

$$\frac{H}{\tau} \cdot \frac{dh^*}{dt^*} = \frac{g\cancel{H}}{8\pi\nu \cancel{A}} \cdot \frac{\cancel{A^2}}{\cancel{H}} \cdot -\frac{h^*(t)}{A^*(h)} \left[\int_0^{h^*} \frac{dz^*}{A^{*2}(z)} \right]^{-1}$$

$$\therefore \tau \sim \frac{\nu H}{gA}$$

- Plug the above expression for τ into $\frac{A}{\nu\tau}$:

$$\frac{A}{\nu\tau} = \frac{A}{\nu} \cdot \frac{gA}{\nu H} = \underbrace{\frac{A}{H^2}}_{\text{small}} \cdot \underbrace{\frac{gAH}{\nu^2}}_{?}$$

We need a scale to relate gravitational effects to viscous.

$\Rightarrow$ Rescale (8.3c-3)!!

$$\underset{\text{characteristic velocity}}{V}A \sim Q_0 = \frac{A^2}{8\pi\mu} \underbrace{\left(\rho g + \frac{\partial P}{\partial z}\right)}_{\text{of the same order}} \sim \frac{gA^2}{\nu}$$

$$\therefore \frac{gA}{\nu} \sim V$$

$$\Rightarrow \frac{A}{\nu\tau} = \frac{A}{H^2} \cdot \frac{VH}{\nu} = \boxed{Re_H \cdot \frac{A}{H^2} \ll 1}$$

This is consistent with our initial inertia-free assumption!