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Fracture Paths from Front Kinetics: Relaxation and Rate Independence

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Abstract

Crack fronts play a fundamental role in engineering models for fracture: they are the location of both crack growth and the energy dissipation due to growth. However, there has not been a rigorous mathematical definition of crack front, nor rigorous mathematical analysis predicting fracture paths using these fronts as the location of growth and dissipation. Here, we give a natural weak definition of crack front and front speed, and consider models of crack growth in which the energy dissipation is a function of the front speed, that is, the dissipation rate at time t is of the form

$$\int_{F(t)} \psi(v(x,t)) \mathrm{d}\mathcal{H}^{N-2}(x)$$

where $F(t)$ is the front at time t and v is the front speed. We show how this dissipation can be used within existing models of quasi-static fracture, as well as in the new dissipation functionals of Mielke–Ortiz. An example of a constrained problem for which there is existence is shown, but in general, if there are no constraints or other energy penalties, this dissipation must be relaxed. We prove a general relaxation formula that gives the surprising result that the effective dissipation is always rate-independent.

1. Introduction

Even when cracks propagate by cleavage, that is, by the breaking of atomic bonds in an otherwise perfect crystal, fracture is best understood as an irreversible and dissipative process. Thus, when an elastic body undergoes fracture, the work stored as elastic energy in the body is less than the work input into the body by the applied loads. The excess work is invested as surface energy on the newly created crack flanks and, from the standpoint of the interior of the elastic body, with surface excluded, is dissipated. Continuum thermodynamics provides a useful framework for describing this irreversibility attendant to fracture. Thus, in that framework fracture entails a certain entropy production and, contrariwise, crack healing entails an entropy loss in violation of the second law. In this manner, the dissipation inequality introduces an irreversibility constraint, namely, that the crack area must be an increasing function of time.

Considerable effort has recently been devoted, with notable advances along the way, to developing a mathematically rigorous theory of fracture within the framework of the modern calculus of variations. This framework, based on the space of special functions of bounded variation SBV and Ambrosio's SBV compactness theorem [1], was originally developed primarily for the study of energy minimization problems; therefore, its application to fracture evolution requires additional development in order to properly account for the no-healing irreversibility constraint in models for crack evolution. The approach so far to the mathematical analysis of rate-independent fracture processes consists of the minimization of incremental energy functionals that geometrically or energetically constrain crack increments in order to enforce irreversibility, and then taking the limit as the time-step goes to zero [4,6,7]. The spaces *SBV* and *SBD* (special functions of bounded deformation) supply a powerful functional foundation for the development of the theory. In particular, they provide an efficient accounting device, the singular or jump set, for describing the crack surface.

In this paper we depart from this—by now standard—paradigm and consider crack trajectories, as well as regard fracture as an irreversible process, ab initio. Thus, we regard the body as a dissipative system in which the dissipation is concentrated at the crack front. In addition, crack advance is governed by a kinetic equation, the so-called crack-front equation of motion, which relates the front velocity to the energetic driving force. In this manner, physically important fracture phenomena, not necessarily rate-independent, such as Paris-law fatigue crack growth [10] and dynamic crack growth (for example, [11]) can potentially be properly accounted for.

Evidently, in the present approach the crack front emerges as a central object for study. Interestingly, whereas the singular or jump set of SBV or SBD functions has been extensively studied, the crack front, a set of co-dimension two, has much less mathematical support. One of the objectives of this paper is to initiate the mathematical study of crack fronts. In particular, we give a natural weak definition of crack front and front speed, and consider models of crack growth where the energy dissipation occurs at the crack front and is a nonlinear function of the front speed, so that it would seem that these models cannot be reformulated without reference to the fronts.

In order to couch the resulting evolution problem within the framework of the calculus of variations, we resort to a class of variational principles recently proposed by Mielke and Ortiz [9]. These variational principles are tailored to dissipative systems and are predicated on energy-dissipation functionals whose minimization returns entire trajectories of the system. We define front for a certain class of these trajectories, and formulate our model within this energy-dissipation framework.

Specifically, we consider the class of trajectories u with corresponding *crack trajectory* C that is increasing and such that at each time t the discontinuity set

$S(u(t))$ is a subset of $C(t)$ (up to a set of $\mathcal{H}^{N-1}$ measure zero). Furthermore, the crack trajectory has a *front representation*, that is, there exists a function $F : [0, T] \to 2^{\Omega}$, and a family of functions $v(\cdot, t) : F(t) \to \mathbb{R}$, such that

$$\int_0^T \dot{\varphi}(t) \int_{C(t)} f(x) \mathrm{d}\mathcal{H}^{N-1}(x)\, \mathrm{d}t = -\int_0^T \varphi(t) \int_{F(t)} f(x) v(x, t) \mathrm{d}\mathcal{H}^{N-2}(x)\, \mathrm{d}t \tag{1}$$

$$\forall \varphi \in C_0^1([0, T]), \forall f \in C_0(\Omega')$$

where $\Omega \subset\subset \Omega'$. We call the set $F(t)$ the *crack front* or *front* at time t, and v the *front speed*. Note that if (u, C) satisfies (1) then necessarily the measure of C is absolutely continuous in time; however, it is unclear if absolute continuity is sufficient. A quick calculation also shows that restricting to trajectories with $v \geqq 0$ provides a new and equivalent way of enforcing the irreversibility of fracture, that is, the monotonicity of C.

With this class we can then consider the problem of minimizing energies of the form (see [9])

$$I_\epsilon[u] := \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \left\{ \frac{1}{\epsilon} \int_\Omega W(\nabla u(x, t)) \mathrm{d}x + \int_{F(t)} \psi(v(x, t)) \mathrm{d}\mathcal{H}^{N-2}(x) \right\} \mathrm{d}t, \tag{2}$$

where $\epsilon > 0$ is fixed. In Section 2 below, we discuss rate problems that can be written in this form, while in Section 3 we justify this functional for fracture specifically.

A critical fact about this class of trajectories is that in order for a minimizing sequence $\{u_i\}_{i=1}^{\infty}$ of (2) to converge (in the natural sense, to be described later) to a trajectory u with corresponding crack C having a front representation, it is necessary that ψ have superlinear growth at infinity, but this is *not sufficient*. There are two reasons for this lack of compactness. First, it is possible that the discontinuity sets of the u_i close up as $i \to \infty$ only for t within some time interval, so that the limit u has discontinuity sets that appear instantaneously at the end of this interval. Second, these sequences can have crack sets that exhibit the onset of a *mother–daughter* microstructure, meaning that the crack grows by creating many small cracks just ahead of the macroscopic crack front, effectively bypassing the superlinear growth of ψ.

Our approach to the first issue is a weakening of the natural choice of C for a given trajectory u—that $C(t)$ is the smallest crack set containing all prior discontinuities of u. Instead, we only require the inclusion of discontinuity sets, namely, that up to sets of $\mathcal{H}^{N-1}$ measure zero,

$$S(u(\tau)) \subset C(t) \quad \forall \tau \in [0, t].$$

We will present two approaches to the second issue, organized in this paper as follows. In Section 5, we will constrain the admissible trajectories to prevent mother-daughter type microstructures and ensure compactness of our constrained class. The corresponding variational problem is analyzed in a two dimensional setting, finally showing the existence of an optimal crack path (Theorem 5.2). In Section 6 we allow such microstructures generally, in N dimensions and without constraints on admissible trajectories, which requires relaxation. We will show that the mother–daughter mechanism is only part of the picture, and in fact minimizing sequences will employ a front microstructure that enables them to move at an energetically optimal front speed, which *depends only on the function* ψ. We thereby show that, remarkably, any energy whose dissipation rate is of the form

$$\int_{F(t)} \psi(v) \mathrm{d}\mathcal{H}^{N-2}$$

relaxes to an energy whose dissipation rate is proportional to the front speed, that is, a rate-independent dissipation, and so also a Griffith energy dissipation (Theorem 6.13).

Perhaps the most natural example for which we would not have expected relaxation to a rate-independent dissipation is $\psi(v) = \alpha + v^p$, giving the energy

$$I_\epsilon[u, C] = \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \left\{ \frac{1}{\epsilon} \int_\Omega W(\nabla u(x,t)) \mathrm{d}x + \int_{F(t)} \left(\alpha + v^p(x,t)\right) \mathrm{d}\mathcal{H}^{N-2}(x) \right\} \mathrm{d}t, \tag{3}$$

with $\alpha > 0$ and $p > 1$. While it would seem that having a fixed penalty on the front size and a superlinear penalty on the front speed would prevent microstructure, let alone relaxation to rate-independence, the relaxation result of Theorem 6.13 shows that this is not the case.

We conclude with two remarks about our results. First, we note that this front-based approach can be incorporated into the discrete time, crack increment formulation for quasi-static crack growth (see Remark 3.1 below). Also, our relaxation proof is unnecessarily strong, in the sense that given any trajectory (u, C), we build optimal approximations (u_n, C_n) such that for a sequence of discrete times $\{t_i^n\}$ with $(t_{i+1}^n - t_i^n) \to 0$, $C(t_i^n) \subset C_n(t) \subset C(t_{i+1}^n)$ for $t \in [t_i^n, t_{i+1}^n]$. Similarly, $u_n(t) = u(t_i^n)$ for $t \in [t_i^n, t_{i+1}^n)$.

2. Minimum principles for rate problems in mechanics

Many physical systems are governed by problems of the *rate form*. Thus, let $u \in Y$ be a field that describes the state of the system, where Y is the corresponding configuration space. For the systems under consideration, the trajectory $u : (0, T) \to Y$ over a time interval $(0, T)$ is governed by the problem:

$$u(0) = u_0 \tag{4a}$$

$$\dot{u}(t) = v(t) \tag{4b}$$

$$v(t) \in \operatorname{argmin}\{G(t, u(t), v(t))\} \tag{4c}$$

where $\dot{u}(t)$ is the time derivative, or *rate*, of u at time t; $u_0 \in Y$ is the initial state of the system; and $G : (0, T) \times Y \times Y \to \bar{\mathbb{R}}$ is a rate functional. Problem (4) entails a sequence of minimum problems parameterized by time. For every time, the minimum problem (4c), or *rate problem*, returns the rate $v(t)$ corresponding to the known state $u(t)$. Integration of these rates in time then determines the evolution of the system.

A special example of rate problem (4c) arises in evolutionary problems governed by rate equations of the form

$$0 \in \partial\Psi(\dot{u}(t)) + DE(t, u(t)), \tag{5a}$$

$$u(0) = u_0, \tag{5b}$$

where $\Psi : Y \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ is a convex dissipation potential; $E : Y \to \mathbb{R}_\infty$ is an energy function; $\partial\Psi$ is the subdifferential of Ψ, representing the system of dissipative forces; DE is the Fréchet derivative of E, representing the conservative force system; and time t varies in the interval $[0, T]$. Equation (5a) establishes a balance between dissipative forces and conservative forces, and the trajectory $u(t)$ of the system is the result of this balance and of the initial condition (5b). In this particular case, the rate functional takes the *additive* form

$$G(t, u(t), v(t)) = \Psi(v(t)) + DE(t, u(t))v(t). \tag{6}$$

Whereas for fixed time the rate of evolution of the system is characterized variationally by the rate problem (4c), the trajectories of the system lack an obvious variational characterization. Specifically, the lack of a minimum principle of trajectories forestalls the application of relaxation, gamma convergence, and other methods of the calculus of variations to the determination of the effective energetics and kinetics of systems exhibiting evolving microstructures.

Mielke and Ortiz [9] have proposed a class of variational principles for trajectories that addresses this difficulty. The fundamental idea is to *string together* the minimum problems (4c) for different times into a single minimum principle. In order to ensure causality, the rate problems corresponding to earlier times are given overwhelmingly more weight than the rate problems corresponding to later times. This leads to the consideration of the family of functionals

$$F_\epsilon(u) = \int_0^T \mathrm{e}^{-t/\epsilon} G(t, u(t), \dot{u}(t))\, \mathrm{d}t \tag{7}$$

and to the minimum principles

$$\inf_{u \in X} F_\epsilon(u) \tag{8}$$

where X is a space of functions from $(0, T)$ to Y, or *trajectories*, such that $u(0) = u_0$. We shall refer to F_ϵ as the *energy-dissipation functional* to acknowledge the fact that F_ϵ accounts for both the energetics and the dissipation characteristics of the system. For additive problems of the form (5), an alternative form of the energy dissipation functional can be obtained through an integration by parts of the dissipation term, with the result

$$F_\epsilon(u) = \int_0^T \mathrm{e}^{-t/\epsilon} \left[\Psi(\dot{u}) + \frac{1}{\epsilon} E(u)\right] \mathrm{d}t \tag{9}$$

up to inconsequential additive constants.

That the *causal limit* $\epsilon \to 0$ of (8) is equivalent to problem (4) can be established formally from the Euler–Lagrange equations of F_ϵ. Thus, the Euler–Lagrange equation of (4c) is, simply,

$$\partial_v G(t, u, v) = 0 \tag{10}$$

whereas the Euler–Lagrange equations of (8) are:

$$\partial_{\dot{u}} G(t, u(t), \dot{u}(t)) + \epsilon \left\{ \partial_u G(t, u(t), \dot{u}(t)) - \frac{\mathrm{d}}{\mathrm{d}t} \partial_{\dot{u}} G(t, u(t), \dot{u}(t)) \right\} = 0 \tag{11}$$

A comparison of (10) and (11) reveals that, disregarding higher-order terms in ϵ, the minimizers $u(t)$ of (8) are such that $\dot{u}(t)$ solves the rate problem (4c) at all times. The Euler–Lagrange equation (11) may also be regarded as an *elliptic regularization* of problem (4) [9]. Thus, depending on the size of ϵ the system is allowed to *peep* into the future to a greater or lesser extent. In the same manner as the term *rate problem* is used to denote the problem that determines rates, namely problem (4c), we shall use the term *trajectory problem* to refer to the problem that determines the trajectories of the systems, namely problem (8).

A class of problems that is amenable to effective analysis concerns *rate-independent systems* for which the dissipation potential Ψ is homogeneous of degree 1 [9]. A striking first property of rate-independent problems is that all minimizers u^ϵ of F_ϵ satisfy energy balance independently of the value of ϵ. Under suitable coercivity assumptions it is then possible to derive a priori bounds for u^ϵ which likewise are independent of ϵ, with the result that it is possible to extract convergent subsequences and find limiting functions u. Under certain regularity assumptions, it follows that all such limits satisfy the *energetic formulation* of Mielke et al. (see, for example, the survey [8] and references therein) for rate-independent systems of the form (5). Moreover, if $(\Psi_k)_{k \in \mathcal{N}}$ converges to Ψ and E_k Γ-converges to E with respect to appropriate topologies, then the accumulation points of the family $(u_{\epsilon,k})_{\epsilon>0, k\in\mathcal{N}}$ for ϵ, $1/k \to 0$ solve the associated limiting energetic formulation. These results for rate-independent systems provide a first indication that the variational program outlined above indeed works, that is, that the minimizers of the energy-dissipation functionals F_ϵ converge towards trajectories of the evolutionary problem. The case of a general rate functional G remains open at present.

3. Fracture mechanics as a rate problem

Fracture is irreversible, dissipative and is driven by energetic driving forces, which suggests that it should be describable within the energy-dissipation framework outlined in the preceding section. However, whereas the energy of a body undergoing fracture is simply given by its elastic energy, the dissipation attendant to crack growth is concentrated on the crack front and its proper accounting requires carefully crafted measure-theoretical tools. Before embarking on the development of those tools, we begin by briefly recounting the elements of formal fracture mechanics that lead to the formulation of dissipation potentials for growing cracks. We therefore proceed formally and assume regularity and smoothness as required.

We consider an elastic body occupying a domain $\Omega \subset \mathbb{R}^N$, $N \geqq 2$. The boundary $\partial\Omega$ of the body consists of an exterior boundary Γ, corresponding to the boundary of the uncracked body, and a collection of cracks jointly defining a crack set C. In addition, Γ partitions in the usual manner into a displacement boundary Γ_1 and a traction boundary Γ_2. The body undergoes deformations under the action of body forces, displacements prescribed over Γ_1 and tractions applied over Γ_2. Under these conditions, the elastic energy of the body is

$$E(u) = \int_{\Omega} W(x, u, \nabla u)\mathrm{d}x + \int_{\Gamma_2} V(x, u)\,\mathrm{d}\mathcal{H}^{N-1} \tag{12}$$

where dx is the N-dimensional Lebesgue measure, $\mathcal{H}^d$ is the d-dimensional Hausdorff measure, W is the elastic strain energy density of the body and V is the potential of the applied tractions. Suppose now that the applied loads and prescribed displacements are incremented over the time interval $[t, t+\Delta t]$ and that, in response to this incremental loading, the crack set extends from $C(t)$ to $C(t+\Delta t)$. Owing to the irreversibility of fracture, we must necessarily have that $C(t) \subset C(t+\Delta t)$. The elastic energy released during the time increment is

$$\begin{aligned} -\Delta E = &\left[\int_{\Omega} W\left(x, u(t), \nabla u(t)\right)\,\mathrm{d}x + \int_{\Gamma_2} V\left(x, u(t)\right)\,\mathrm{d}\mathcal{H}^{N-1}\right] \\ &- \left[\int_{\Omega} W\left(x, u\left(t+\Delta t\right), \nabla u(t+\Delta t)\right)\,\mathrm{d}x \right. \\ &\left. + \int_{\Gamma_2} V\left(x, u(t+\Delta t)\right)\,\mathrm{d}\mathcal{H}^{N-1}\right]. \end{aligned} \tag{13}$$

Expanding to first order in all incremental terms, we obtain

$$-\Delta E \sim -\left[\int_{\Omega} (\partial_u W \cdot \Delta u + \partial_{\nabla u} W \cdot \nabla \Delta u)\,\mathrm{d}x + \int_{\Gamma_2} \partial_u V \cdot \Delta u\,\mathrm{d}\mathcal{H}^{N-1}\right]. \tag{14}$$

Integrating by parts and using the equations of equilibrium, we find that this expression reduces to

$$-\Delta E \sim \int_{\Delta C} T(t) \cdot [\![u(t+\Delta t)]\!]\,\mathrm{d}\mathcal{H}^{N-1} \tag{15}$$

where

$$T = \partial_{\nabla u} W(x, u, \nabla u)\, n \tag{16}$$

are the internal tractions, with n the unit outward normal to the boundary, and we write $\Delta C = C(t+\Delta t)\backslash C(t)$, Fig. 1a. The corresponding energy release rate now follows as

$$-\dot{E} = -\lim_{\Delta t \to 0} \frac{\Delta E}{\Delta t} = \int_F f(n) v\,\mathrm{d}\mathcal{H}^{N-2} \tag{17}$$

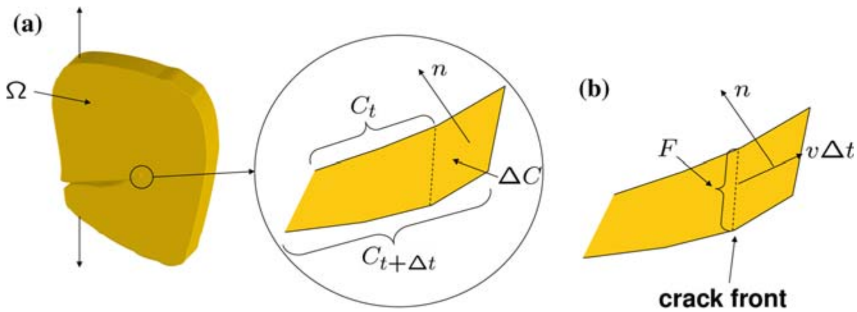

Fig. 1. **a** Crack advancing in a body occupying domain Ω and zoom of the crack-front region showing crack set C_t at time t, contained with crack set $C_{t+\Delta t}$ at time $t + \Delta t$, during which interval of time the crack front sweeps an area ΔC of unit normal n. **b** Detail of advancing front and definition of front velocity

where F is the crack front, Fig. 1b, v is the crack-front velocity

$$f(n) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} (\partial_{\nabla u} W \, n) \cdot [\![u_{t+\Delta t}]\!] \tag{18}$$

is the energetic force acting on the crack front. The identity (17) expresses the rate at which energy flows to—and is subsequently dissipated at—the crack front. In particular, the duality-pairing structure of (17) is conventionally taken to mean that the energetic force $f(n)$ does power, or *drives* on the crack-front velocity v. On this basis, it is customary in fracture mechanics to postulate the existence of a *crack-tip equation of motion* of the form

$$f = \partial\psi(v) \tag{19}$$

where ψ is a dissipation potential density per unit crack-front length. The total dissipation potential for the entire crack front finally follows by additivity as

$$\Psi(v) = \int_F \psi(v) \, d\mathcal{H}^{N-2} \tag{20}$$

We note that constitutive relations of the form (19) can also be derived—instead of just postulated—from (17) and the first and second laws of thermodynamics using Coleman and Noll's method [3]. The crack-tip equation of motion (19) is subject to the dissipation inequality

$$f \cdot v \geqq 0 \tag{21}$$

which follows as a consequence of the second law of thermodynamics. In the present context, the dissipation inequality introduces a unilateral constraint that prevents crack healing.

We note that the dissipation attendant to crack growth is localized to the crack front F, which is a set of co-dimension 2. This is in contrast to energetic theories of fracture based on the SBV or SBD formalisms in which the principal singular set of interest, namely, the crack set, has co-dimension 1. In geometrical measure theory

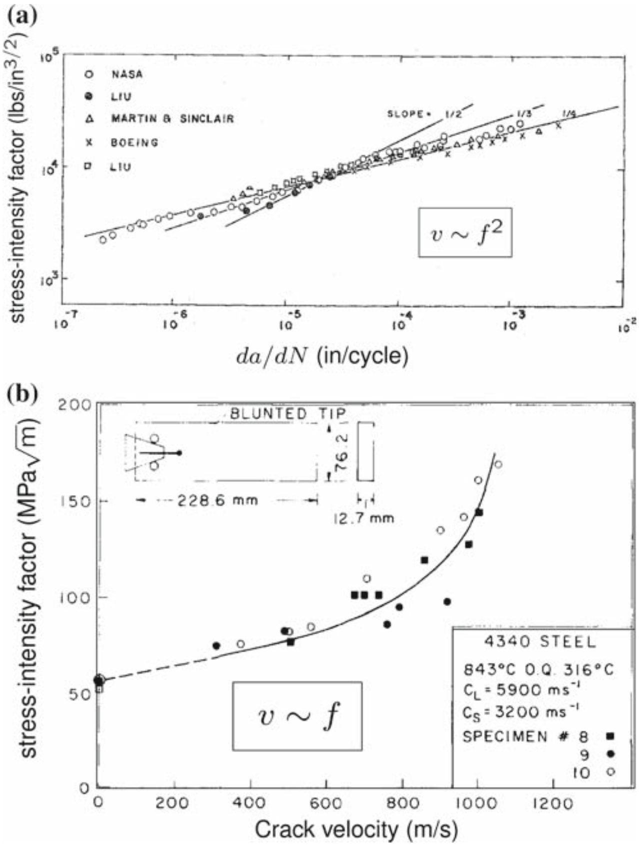

Fig. 2. a Compilation of fatigue data for 2024-T3 aluminum alloy [10]. **b** Dynamic fracture data for 4340 steel [11]. The driving force f scales as the square of the stress-intensity factor. By plotting the driving force versus crack-tip velocity on log–log axes, we find that all the data points collapse on master curves

the structure and properties of sets of co-dimension 2 is less well understood than those of sets of co-dimension 1, which adds difficulty to the energy-dissipation version of fracture mechanics. We also note that in rate-independent theories of fracture mechanics the dissipation is described by a surface energy on the crack flanks and lumped together with the energy.

The observational record lends support to the assumption that crack growth obeys a crack-tip equation of motion of the form (19). By way of example, Fig. 2 shows a compilation of fatigue data for 2024-T3 aluminum alloy from the classical work of Paris and Erdogan [10] and dynamic fracture data for 4340 steel [11]. In the case of fatigue, the number N of loading cycles plays the role of time. In interpreting these data it should also be recalled that in linear-elastic fracture mechanics the driving force f scales as the square of the stress-intensity factor. By plotting the driving force versus crack-tip velocity on log–log axes, one finds that all the data points ostensibly collapse on master curves, suggesting the existence of a crack-tip

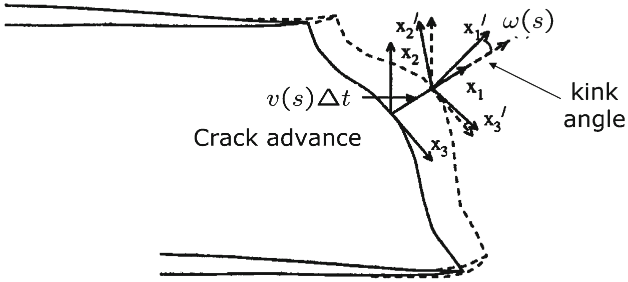

Fig. 3. Local view of the geometry and kinetics of crack advance

equation of motion. The data displayed in Fig. 2 are also suggestive of power-law behavior, possibly with a threshold on the driving force. Thus, with the direction of advance prescribed, for example, by symmetry, the component of the crack-tip equation of motion normal to the front within the tangent plane to the crack takes the form

$$v = C(f - f_0)^m \tag{22}$$

where the threshold $f_0 \geqq 0$, C and m are material constants. If the rate of dissipation is further assumed to be independent of the direction of crack advance, then the dissipation potential follows as

$$\psi(v) = f_0|v| + \frac{mC}{m+1}|v|^{1+1/m} \tag{23}$$

We are now in a position to formulate the *rate problem* (5) for fracture mechanics. In view of identity (17), the rate problem of fracture mechanics reduces to

$$\inf_{v,n} \int_F [\psi(v) - f(n) \cdot v] \, d\mathcal{H}^{N-2} \tag{24}$$

and the corresponding Euler–Lagrange equations are

$$\partial\psi(v) = f(n) \tag{25a}$$

$$\partial\psi^*(f(n)) = 0 \tag{25b}$$

which jointly determine the crack-tip velocity v and direction of advance n. The resulting geometry and kinetics of crack advance are illustrated in Fig. 3, that represents a local neighborhood of the crack front, for example, parametrized by its arc length s, with the local crack geometric described by orthonormal axes tangent to the crack and its front. Because of the constraint $C(t) \subset C(t + \Delta t)$, it follows that the direction of crack advance can locally be described by means of a single *kinking angle* $\omega(s)$. Also, because of the constraint (25b) reduces to one single equation for the determination of $\omega(s)$. We note from (24) that the resulting kinking angle maximizes the energy-release rate or, equivalently, the rate of dissipation $f(n) \cdot v$, and thus we can regard (24) variously as a maximum energy-release or a maximum dissipation principle. Once $\omega(s)$, and by extension $n(s)$, is determined from (25b)

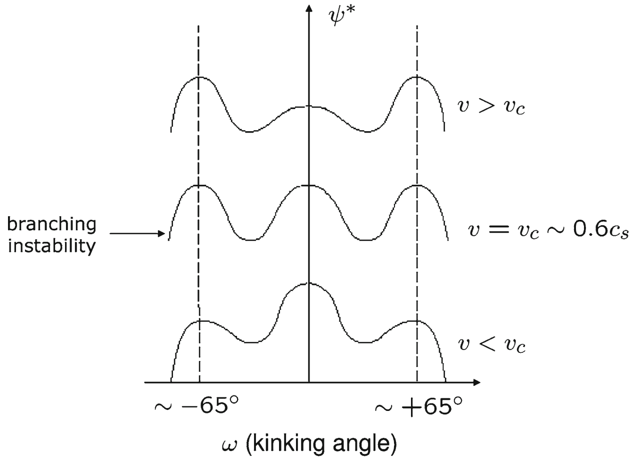

Fig. 4. Dual dissipation density as a function of kinking angle for steady-state dynamic crack growth at different crack tip velocities. The dual energy-dissipation density has a single maximum below a critical crack-tip velocity, corresponding to straight-ahead growth, and two maxima above the critical velocity, corresponding to crack branching [12]

the local crack-front velocity $v(s)$, giving the rate of extension of the crack, follows from (25a), which simply restates the crack-tip equation of motion (Fig. 4).

The energy-dissipation functional (24) can exhibit complex behavior. A case in point is furnished by a dynamic two-dimensional crack propagating in a steady state. In this case, an equivalent static problem can be obtained by introducing a reference frame that moves with the crack tip, and the equivalent static problem thus defined can be analyzed within the energy-dissipation framework just outlined. A classical solution of Yoffe [12] then shows that for crack-tip velocities below a certain critical speed v_c of the order of 60% of the shear wave speed (25b) has a single solution and the crack runs straight ahead. By way of sharp contrast, above the critical speed (25b) has two symmetrical solutions corresponding to kinking angles of the order of $\pm 65°$ corresponding to *crack branching*. In the present variational framework, this classical branching instability of dynamic fracture can thus be understood as a consequence of the lack of convexity of the rate problem, which furnishes a new insight into the phenomenon and opens opportunities for the analysis of crack branching.

On the basis of preceding description of the energetics and dissipation of fracture, we can now exhibit the energy dissipation functional (9) of fracture mechanics, namely,

$$F_\epsilon(u) = \int_0^T \mathrm{e}^{-t/\epsilon} \Bigg[\int_F \psi(v)\, \mathrm{d}\mathcal{H}^{N-2} + \frac{1}{\epsilon} \left(\int_\Omega W(x, u, \nabla u)\, \mathrm{d}x + \int_{\Gamma_2} V(x, u)\, \mathrm{d}\mathcal{H}^{N-1} \right) \Bigg] \mathrm{d}t. \tag{26}$$

Minimization of this energy-dissipation functional supplies the entire crack-path over the time interval $[0, T]$ and the attendant trajectory of the displacement field. The energy-dissipation functional (26) forms the basis of the analysis presented in the remainder of the paper.

Remark 3.1. We close this section by noting that this front-based variational model can also be used in the discrete-time incremental approach, by considering for crack increments ΔC in the time interval $[t_1, t_2]$ the crack energy

$$\inf\left\{\liminf_{n\to\infty}\int_{t_1}^{t_2}\int_{F_n}\psi(v_n)\,\mathrm{d}\mathcal{H}^{N-2}\,\mathrm{d}t\,:\,C_n\to\Delta C\right\}$$

where F_n is the front corresponding to C_n and the convergence $C_n \to \Delta C$ is in the sense described in Section 6. Remarkably, as a consequence of the results in that section (see Remark 6.11 and Theorem 6.12), this inf is simply

$$\inf\left\{\liminf_{n\to\infty}\int_{t_1}^{t_2}\int_{F_n}\mathscr{C}v_n\,\mathrm{d}\mathcal{H}^{N-2}\mathrm{d}t\,:\,C_n\to\Delta C\right\}=\mathscr{C}\mathcal{H}^{N-1}(\Delta C),\qquad(27)$$

where

$$\mathscr{C}:=\inf_{s\in(0,\infty)}\frac{\psi(s)}{s}.$$

4. Notation and mathematical setting

We first introduce some notation to be used throughout the paper, which is consistent with [5].

- Ω, a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary, represents the reference configuration of the body. As a mechanism for enforcing boundary conditions (see for instance [6]), Ω' will denote a bounded open set with Lipschitz boundary such that $\Omega\subset\subset\Omega'$.
- For $y\in\mathbb{R}^N$, let $(y^1,\ldots,y^N)$ denote the components of y.
- For $n=0,\ldots,N$ $\mathcal{L}^n$ is the n-dimensional Lebesgue measure and $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure.
- $SBV(\Omega)$ is the space of *special functions of bounded variation on* Ω. For $u\in SBV(\Omega)$, we will denote the approximate discontinuity set of u as $S(u)$ (see [2]). $SBV_p(\Omega)$ will denote those $u\in SBV(\Omega)$ such that $\nabla u\in L^p(\Omega)$.
- We will say that a sequence $\{v_n\}_{n=1}^\infty\subset SBV(\Omega)$ converges to $v\in SBV(\Omega)$ (or $v_n\overset{SBV}{\to}v$) if

$$\begin{cases}\nabla v_n\rightharpoonup\nabla v \text{ in } L^1(\Omega);\\ [v_n]\nu_n\mathcal{H}^{N-1}\lfloor S(v_n)\overset{*}{\rightharpoonup}[v]\nu\mathcal{H}^{N-1}\lfloor S(v)\text{ as measures;}\\ v_n\to v\text{ in } L^1(\Omega);\ \text{and}\\ v_n\overset{*}{\rightharpoonup}v\text{ in } L^\infty(\Omega),\end{cases}$$

where ν denotes the normal to $S(v)$, and $[v]$ the jump of v. Note that, as a consequence (see [1]),

$$\mathcal{H}^{N-1}(S(v)) \leqq \liminf_{n\to\infty} \mathcal{H}^{N-1}(S(v_n)) \tag{28}$$

whenever $v_n \overset{SBV}{\to} v$.

- For any set of finite perimeter E, $\partial^* E$ denotes the reduced boundary of E and for $x \in \partial^* E$ $\nu_E(x)$ denotes the measure theoretic outer normal to E at x.
- For $\xi \in \mathbb{R}$, let E_ξ^w denote the ξ super level set of w, that is, $E_\xi^w := \{x \in \Omega : w(x) > \xi\}$.
- For $\{K_i\}_{i=1}^\infty$, $K_i \subset \mathbb{R}^2$, we use the notation $K = \operatorname*{H-lim}_{i\to\infty} K_i$ or $K_i \overset{H}{\to} K$ to mean that K_i converges to K in the Hausdorff metric.
- $A \overset{\sim}{\subset} B$ means that $\mathcal{H}^{N-1}(A\backslash B) = 0$. $A \overset{\sim}{=} B$ means $\mathcal{H}^{N-1}(A \triangle B) = 0$.
- 2^X denotes the power set of X.
- $Q(x, r)$ is a cube in $\mathbb{R}^N$ centered at x with side length $2r$.
- $B(x, r)$ is a closed ball centered at x with radius r.
- $W : \mathbb{R}^N \to \mathbb{R}$ is convex with minimum attained for $\xi \in \mathbb{R}^N$ with $\|\xi\|_{\mathbb{R}^N} = 0$ and satisfies $C_1|\xi|^p - \dfrac{1}{C_1} \leqq W(\xi) \leqq C_2(|\xi|^p + 1)$ for some positive constants C_1, C_2 and some $p > 1$.

5. Existence for constrained trajectories

In this section, we present an existence result for a constrained version of the problem that we introduced in Section 1. We are restricting our consideration to the two dimensional case ($\Omega \subset \mathbb{R}^2$), and, motivated by the compactness issues for the class of trajectories that satisfy (1) (see Section 1), we define a class of *constrained* trajectories:

Definition 5.1. For fixed $p' > 0$, the class $\mathcal{T}_{p'}$ is the set of triples (u, C, F) such that:

1. u satisfies:
   (a) $u(\cdot, t) \in SBV_p(\Omega') \ \forall t \in [0, T]$
   (b) $\int_\Omega W(\nabla u(x, \cdot))\mathrm{d}x \in L^1([0, T]; \mathbb{R})$
   (c) $\forall t \in [0, T]$, $u(\cdot, t) = g$ on $\Omega' \backslash \bar{\Omega}$, where $g \in L^\infty(\Omega') \cap H^1(\Omega')$ is given.
2. $C : [0, T] \to \left\{K \subset \bar{\Omega} : K \text{ is } \mathcal{H}^1 \text{ measurable}, \mathcal{H}^1(K) < \infty\right\}$ such that:
   (a) $C(0) \overset{\sim}{=} C_0$, for given C_0
   (b) C nondecreasing: $\forall \tau < t$, $C(\tau) \overset{\sim}{\subset} C(t)$
   (c) $\forall t \in [0, T]$, $S(u(t)) \overset{\sim}{\subset} C(t)$
   (d) $F \in W^{1,p'}([0, T]; \bar{\Omega})$, and there exists a family of functions $v(\cdot, t) : F(t) \to \mathbb{R}$, such that

$$\int_0^T \dot{\varphi}(t) \int_{C(t)} f(x)\mathrm{d}\mathcal{H}^1(x)\mathrm{d}t = -\int_0^T \varphi(t) \int_{F(t)} f(x)v(x, t)\mathrm{d}\mathcal{H}^0(x)\mathrm{d}t$$

$\forall \varphi \in C_0^1([0, T]), \forall f \in C_0(\Omega')$.

Property 2 expresses the fact that we are considering a relaxed definition of crack set, as discussed in the introduction. By Property 2d, we are only considering those trajectories that satisfy the front representation, and further that their fronts are at most one point $\forall t \in [0, T]$ with no jumps in the position of this front. Since the class of trajectories that have a one point front moving continuously is not closed, we allow the front point to move inside of the existing crack set (with $v = 0$). Therefore, we can choose $F \in W^{1,p'}([0,T], \bar{\Omega})$ such that at every $t \in [0,T]$, the front at time t is a subset of $F(t)$. We will consider a dissipation potential of a similar character to (23), in particular we require superlinear growth of the dissipation potential. However, since F can move inside of the existing crack set, a sequence $\{q_i\}_{i=1}^{\infty} \subset \mathcal{T}_{p'}$ will have a subsequence that converges to an element of $\mathcal{T}_{p'}$ only if $\sup_i \|\dot{F}_i\|_{L^{p'}}$ is bounded; therefore, in order to ensure compactness, we must penalize the derivative of F in the functional. Accordingly, we will minimize

$$I_{\epsilon,p'}[q] := \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \left\{ \frac{1}{\epsilon} \int_\Omega W(\nabla u(x,t))\mathrm{d}x + \int_{F(t)} \left|\dot{F}\right|^{p'}(t)\mathrm{d}\mathcal{H}^0(x) \right\} \mathrm{d}t$$

over $q = (u, C, F) \in \mathcal{T}_{p'}$, where $\epsilon > 0$ and $p' > 1$ are fixed. Note that the proof of Theorem 5.2 below applies to all convex potentials with p' growth, in particular for $|\dot{F}| + |\dot{F}|^{p'}$, as in (23). Since $F(t)$ is only one point, the energy is simply

$$I_{\epsilon,p'}[q] := \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \left\{ \frac{1}{\epsilon} \int_\Omega W(\nabla u(x,t))\mathrm{d}x + \left|\dot{F}\right|^{p'}(t) \right\} \mathrm{d}t.$$

Theorem 5.2. *There exists a minimizer of $I_{\epsilon,p'}$ in $\mathcal{T}_{p'}$.*

Proof. Let $\{q_i\}_{i=1}^{\infty} \subset \mathcal{T}_{p'}$ be a minimizing sequence for $I_{\epsilon,p'}$, meaning

$$\lim_{i\to\infty} I_{\epsilon,p'}[q_i] = \inf_{q\in\mathcal{T}_{p'}} I_{\epsilon,p'}[q].$$

This implies that

$$\sup_i \|\dot{F}_i\|_{L^{p'}([0,T];\mathbb{R}^2)} < \infty. \tag{29}$$

Since $p' > 1$, then by (29), Theorem 1 in Section 4.6 of [5], and Morrey's inequality (Theorem 3 in Section 4.5.3 of [5]) there is an $F \in W^{1,p'}([0,T]; \bar{\Omega})$ such that, up to a subsequence that we will not relabel,

$$F_i \to F \text{ in } L^\infty([0,T]; \bar{\Omega}) \text{ and} \tag{30}$$

$$\dot{F}_i \rightharpoonup \dot{F} \text{ in } L^{p'}([0,T]; \mathbb{R}^2). \tag{31}$$

Note that (31) implies the following:

$$\int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \left|\dot{F}\right|^{p'}(t)\mathrm{d}t \leqq \liminf_{i\to\infty} \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \left|\dot{F}_i\right|^{p'}(t)\mathrm{d}t. \tag{32}$$

Set $C(t) := C_0 \cup \bigcup_{\tau \leqq t} F(\tau)$ and $\tilde{C}_i(t) := C_0 \cup \bigcup_{\tau \leqq t} F_i(\tau)$. Since $F_i \to F$ uniformly then $\forall t \in [0, T]$ $\tilde{C}_i(t) \overset{H}{\to} C(t)$. Construct $u : \Omega \times [0, T] \to \mathbb{R}$ as follows: $\forall t \in [0, T]$, take

$$u(\cdot, t) \in \operatorname{argmin} \left\{ \int_\Omega W(\nabla z) \mathrm{d}x : z \in SBV(\Omega), \; S(z) \,\tilde{\subset}\, C(t), \; z = g \text{ in } \Omega' \backslash \Omega \right\}, \tag{33}$$

which is nonempty by the properties of W and the compactness of the space SBV (Theorems 4.7 and 4.8 of [2]).
Let $q := (u, C, F)$ as defined above. We will now show that $q \in \mathcal{T}_{p'}$, and that it is a minimizer of $I_{\epsilon, p'}$. First, note that properties 1a, 1c, 2a, 2b, and 2c hold for q by construction. Also, since C is nondecreasing, the map

$$t \mapsto \int_\Omega W(\nabla u(x, t)) \mathrm{d}x$$

is nonincreasing, is continuous almost everywhere and therefore $\mathcal{L}^1$ measurable. This, combined with the lower semicontinuity of the bulk part of the energy means that property 1b is satisfied.

Next, we verify that the pair (C, F) satisfies property 2d of the definition of $\mathcal{T}_{p'}$. Choose a sequence $\{\eta_k\}_{k=1}^\infty \subset C^\infty([0, T]; \bar{\Omega})$ such that $\eta_k \to F$ strongly in $W^{1,p'}([0, T]; \bar{\Omega})$ (see Section 4.2 Theorem 3 of [5]). Then Morrey's inequality (Section 4.5.3 Theorem 3 of [5]) implies

$$\eta_k \to F \text{ strongly in } L^\infty([0, T]; \bar{\Omega}).$$

Let $\Gamma_k(t) := \bigcup_{\tau \leqq t} \eta_k(\tau)$. According to the Area formula (Theorem 1 in Section 3.3.2 in [5]) we have $\forall k \in \mathbb{N}$ and $t < t' \in [0, T]$

$$\begin{aligned} \int_t^{t'} |\dot{\eta}_k| \, \mathrm{d}t &= \int_{\bar{\Omega}} \mathcal{H}^0 \left([t, t'] \cap \eta_k^{-1}(\{y\}) \right) \mathrm{d}\mathcal{H}^1(y) \\ &\geqq \mathcal{H}^1(\Gamma_k(t') \backslash \Gamma_k(t)). \end{aligned} \tag{34}$$

Using the uniform convergence $\eta_k \to F$ (in particular that $\forall t \in [0, T]$ $\Gamma_k(t) \overset{H}{\to} C(t) \backslash C_0$) and the fact that $\Gamma_k(t)$ connected, we have for all $t < t' \in [0, T]$

$$\begin{aligned} \mathcal{H}^1(C(t') \backslash C(t)) &\leqq \liminf_{k \to \infty} \mathcal{H}^1 \left(\Gamma_k(t') \backslash \Gamma_k(t) \right) \\ &\leqq \lim_{k \to \infty} \int_t^{t'} |\dot{\eta}_k| (s) \mathrm{d}s \\ &= \int_t^{t'} \left| \dot{F} \right| (s) \mathrm{d}s. \end{aligned} \tag{35}$$

Using (35), we have that for any $f \in C_0(\Omega')$ and all $t < t' \in [0, T]$,

$$\left|\int_{C(t')} f \mathrm{d}\mathcal{H}^1 - \int_{C(t)} f \mathrm{d}\mathcal{H}^1\right| = \left|\int_{C(t')\setminus C(t)} f \mathrm{d}\mathcal{H}^1\right| \\ \leqq \|f\|_{L^\infty(\Omega')} \mathcal{H}^1(C(t')\setminus C(t)) \\ \leqq \|f\|_{L^\infty(\Omega')} \int_t^{t'} \left|\dot{F}\right|(s)\mathrm{d}s. \tag{36}$$

The estimate (36) means that, for every $f \in C_0(\Omega')$, the map

$$t \mapsto \int_{C(t)} f(x)\mathrm{d}\mathcal{H}^1(x) \tag{37}$$

is absolutely continuous, and so there exists $D_f \in L^1([0, T]; \mathbb{R})$ such that

$$\int_0^T \dot{\varphi}(t) \int_{C(t)} f(x)\mathrm{d}\mathcal{H}^1(x)\mathrm{d}t = -\int_0^T \varphi(t) D_f(t)\mathrm{d}t \tag{38}$$

for all $\varphi \in C_0^1([0, T]; \mathbb{R})$. In particular, taking $f \equiv 1$ in $\bar{\Omega}$ there is a $D \in L^1([0, T]; \mathbb{R})$ such that

$$\int_0^T \dot{\varphi}(t)\mathcal{H}^1(C(t))\mathrm{d}t = -\int_0^T \varphi(t) D(t)\mathrm{d}t$$

for all $\varphi \in C_0^1([0, T]; \mathbb{R})$. Since for any $f \in C_0(\Omega')$ with $\|f\|_{L^\infty(\Omega')} \leqq 1$ the map

$$t \mapsto \mathcal{H}^1(C(t)) - \int_{C(t)} f(x)\mathrm{d}\mathcal{H}^1(x)$$

is nondecreasing, for any $f \in C_0(\Omega')$ one can show that there is a representative of D_f, denoted D_f^*, so that for all $t \in [0, T]$

$$\frac{1}{\|f\|_{L^\infty(\Omega')}} D_f^*(t) \leqq D(t) < \infty$$

and for almost every $t \in [0, T]$ the map $f \to D_f^*(t)$ is a bounded linear map on $C_0(\Omega')$. By the Riesz representation theorem (Theorem 1 in Section 1.8 of [5]), there exists a family of measures $\{\mu_t\}_{t\in[0,T]}$ such that for all $f \in C_0(\Omega')$

$$D_f(t) = \int_\Omega f(x)\mathrm{d}\mu_t(x) \tag{39}$$

at almost every $t \in [0, T]$. Hence,

$$\int_0^T \dot{\varphi}(t) \int_{C(t)} f(x)\mathrm{d}\mathcal{H}^1(x)\,\mathrm{d}t = -\int_0^T \varphi(t) \int_{\Omega'} f(x)\mathrm{d}\mu_t(x)\mathrm{d}t, \tag{40}$$

for all $\varphi \in C_0^1([0, T]; \mathbb{R})$ and $f \in C_0(\Omega'; \mathbb{R})$. Now, we show that for almost every $t \in [0, T]$, the measure μ_t is supported on $F(t)$. Since $F \in W^{1,p'}([0, T]; \bar{\Omega})$,

$p' > 1$, F is uniformly continuous on $[0, T]$. For each $n \in \mathbb{N}$ choose $\delta_n > 0$ so that for $a, b \in [0, T]$ with $|a - b| < \delta_n$, $|F(a) - F(b)| < 1/(2n)$. Fixing an $n \in \mathbb{N}$, choose a finite set of open intervals $\{(a_k, b_k)\}_{k=1}^{z}$ such that

$$0 < |b_k - a_k| < \delta_n \quad \forall k,$$

and

$$\mathcal{L}^1\left([0, T]\setminus \bigcup_k (a_k, b_k)\right) = 0.$$

Fix k and then choose some $t_k \in (a_k, b_k)$. Set

$$B := B(F(t_k), 1/(2n)).$$

For any $t \in (a_k, b_k)$, $C(t)\setminus C(a_k) \subset B$, which means

$$C(t)\setminus B = C(a_k)\setminus B$$

and so for all $f \in C_0(\Omega'\setminus B)$

$$\frac{\mathrm{d}}{\mathrm{d}t}(C(t)\setminus B) = 0$$

for almost every $t \in (a_k, b_k)$. Thus, by (40), for any $f \in C_0(\Omega'\setminus B)$ and almost every $t \in (a_k, b_k)$

$$\int_{\Omega'} f(x)\mathrm{d}\mu_t(x) = 0,$$

and so for almost every $t \in (a_k, b_k)$

$$\mu_t(\Omega'\setminus B) = 0. \tag{41}$$

By the choice of the diameter of B, we know that for every $t \in (a_k, b_k)$

$$B \subset B(F(t), 1/n)$$

and so for almost every $t \in (a_k, b_k)$

$$\begin{aligned}\mu_t\left(\Omega'\setminus B(F(t), 1/n)\right) &\leqq \mu_t\left(\Omega'\setminus B\right)\\ &= 0.\end{aligned}$$

Repeating this argument for each k, and setting

$$G_n := \{t \in [0, T] : \mu_t\left(\Omega\setminus B(F(t), 1/n)\right) > 0\},$$

we have that

$$\mathcal{L}^1(G_n) = 0$$

for all $n \in \mathbb{N}$ and so the set

$$G := \{t \in [0, T] : \mu_t\left(\Omega\setminus F(t)\right) > 0\}$$

has zero measure. This means that for $t \in [0, T]\backslash G$

$$\mu_t << \mathcal{H}^0 \lfloor F(t),$$

and setting

$$v(x,t) := \frac{\mathrm{d}\mu_t}{\mathrm{d}\mathcal{H}^0 \lfloor F(t)}(x)$$

we apply (40) to find

$$\int_0^T \dot{\varphi}(t) \int_{C(t)} f(x) \mathrm{d}\mathcal{H}^1(x) \mathrm{d}t = -\int_0^T \varphi(t) \int_{F(t)} f(x) v(x,t) \mathrm{d}\mathcal{H}^0(x) \mathrm{d}t \tag{42}$$

for all $\varphi \in C_0^1([0,T];\mathbb{R})$ and $f \in C_0(\Omega';\mathbb{R})$. Therefore the triple $q = (u, C, F)$ satisfies property 2d of the definition of $\mathcal{T}_{p'}$. $\square$

It remains only to show the lower semicontinuity of the bulk part of the energy. We will use this claim about our sequence u_i and the C constructed above:

Claim. Suppose that for some $w \in SBV(\Omega)$, $u_i(\cdot, t) \overset{SBV}{\to} w$. Then $S(w) \overset{\sim}{\subset} C(t)$.

Proof of Claim. Recall that $\tilde{C}_i(t) \overset{H}{\to} C(t)$. Now let $x_0 \in \Omega' \backslash C(t)$. Since $C(t)$ is closed, there exists $t^* \in [0, t]$ such that

$$D := \operatorname{dist}(F(t^*), x_0) = \min_{s \in [0,t]} \operatorname{dist}(F(s), x_0)) > 0.$$

Set

$$B := B(x_0, D/2).$$

Then there exists $N \in \mathbb{N}$ such that $\forall i > N$

$$\tilde{C}_i(t) \cap B = \emptyset.$$

By definition of $\tilde{C}_i$, and since for each i the pair (C_i, F_i) satisfies the front representation formula with a front speed v_i, for any $f \in C_0(B)$ and $i > N$

$$\begin{aligned} \int_{C_i(t)} f(x) \mathrm{d}\mathcal{H}^1(x) &= \int_0^t \int_{F_i(s)} f(x) v_i(x,s) \mathrm{d}\mathcal{H}^0(x) \mathrm{d}s \\ &= \int_0^t \int_{F_i(s) \cap B} f(x) v_i(x,s) \mathrm{d}\mathcal{H}^0(x) \mathrm{d}s \\ &= 0. \end{aligned} \tag{43}$$

Then (43) implies

$$\mathcal{H}^1(C_i(t) \cap B) = 0$$

for $i > N$. By property 2c of Definition 5.1, we have

$$\mathcal{H}^1(S(u_i(t)) \cap B) = 0$$

for $i > N$. Therefore, applying (28) with $u_i|_B$ and $w|_B$ we have that

$$\mathcal{H}^1(S(w) \cap B) \leqq \liminf_{i\to\infty} \mathcal{H}^1(S(u_i(t)) \cap B) = 0.$$

Since the above argument holds for any ball with radius less than $D/2$, and since x_0 was arbitrary, this proves the claim.

Now, to show that the bulk energy is lower semicontinuous, fix $t \in [0, T]$. Take a subsequence of $\{u_i\}_{i=1}^{\infty}$ such that:

$$\lim_{k\to\infty} \int_\Omega W(\nabla u_{i_k}(x,t))\mathrm{d}x = \liminf_{i\to\infty} \int_\Omega W(\nabla u_i(x,t))\mathrm{d}x.$$

We can assume, without loss of generality, that $\sup_k \|u_{i_k}(t)\|_{L^\infty} < +\infty$ since truncation merely lowers the elastic energy. By the compactness of the space of SBV (Theorem 4.8 of [2]), there exists $\bar{u}_t \in SBV(\Omega)$ such that, up to a further subsequence that we will not relabel,

$$u_{i_k} \overset{SBV}{\to} \bar{u}_t.$$

By the above claim

$$S(\bar{u}_t) \mathrel{\tilde{\subset}} C(t),$$

and so applying the definition of u (recall (33)), we have

$$\int_\Omega W(\nabla u(x,t))\mathrm{d}x \leqq \int_\Omega W(\nabla \bar{u}_t(x,t))\mathrm{d}x.$$

Therefore,

$$\begin{aligned}\int_\Omega W(\nabla u(x,t))\mathrm{d}x &\leqq \int_\Omega W(\nabla \bar{u}_t(x,t))\mathrm{d}x \\ &\leqq \lim_{k\to\infty} \int_\Omega W(\nabla u_{i_k}(x,t))\mathrm{d}x \\ &= \liminf_{i\to\infty} \int_\Omega W(\nabla u_i(x,t))\mathrm{d}x.\end{aligned}$$

Since the above holds for each $t \in [0, T]$, then the lower bound on W and Fatou's Lemma (see [2]) implies:

$$\begin{aligned}\int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \int_\Omega W(\nabla u(x,t))\mathrm{d}x\,\mathrm{d}t &\leqq \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \left\{\liminf_{i\to\infty} \int_\Omega W(\nabla u_i(x,t))\mathrm{d}x\right\} \mathrm{d}t \\ &\leqq \liminf_{i\to\infty} \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \int_\Omega W(\nabla u_i(x,t))\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{44}$$

Combining (32) and (44) gives

$$I_{\epsilon,p'}[q] \leqq \liminf_{i\to\infty} I_{\epsilon,p'}[q_i],$$

which establishes that the triple $q = (u, C, F)$ is a minimizer of $I_{\epsilon,p'}$. □

6. Relaxation and rate-independence

For energies of the form

$$I_\epsilon[q] := \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \left\{ \frac{1}{\epsilon} \int_\Omega W(\nabla u(x,t))\mathrm{d}x + \int_{F(t)} \psi(v(x,t))\mathrm{d}\mathcal{H}^{N-2}(x) \right\} \mathrm{d}t, \tag{45}$$

(where $q \in \mathcal{T}$, $\epsilon > 0$ is fixed, and $\psi : [0,\infty) \to [0,\infty)$ is continuous) minimizing sequences can exhibit the onset of microstructures that involve the geometry of the crack front, which prevents the existence of a minimizer without strong restrictions on that geometry (see Section 5). In this section we will characterize the optimal crack front microstructure and prove a formula for the relaxation of the dissipation part of energies of the form (45) (see Theorem 6.13). This result holds in any dimension and without *a priori* constraints on the crack fronts. This section is organized as follows. Section 6.1 contains the definition for the appropriate class of fracture trajectories and other definitions useful for the remainder of Section 6. In Section 6.2 we describe the notion of convergence for which we prove the relaxation result- this convergence is extremely weak and thus the result of Theorem 6.13 holds in practical settings. Section 6.3 contains Theorem 6.13 and its proof.

6.1. Definitions

Definition 6.1. The class $\mathcal{T}$ is the set of pairs (u, C) such that:

1. u satisfies:
   (a) $u(\cdot,t) \in SBV_p(\Omega') \; \forall t \in [0,T]$
   (b) $\int_\Omega W(\nabla u(x,\cdot))\mathrm{d}x \in L^1([0,T];\mathbb{R})$
   (c) $\forall t \in [0,T]$, $u(\cdot,t) = g$ on $\Omega' \backslash \bar{\Omega}$, where $g \in L^\infty(\Omega') \cap H^1(\Omega')$ is given
2. $C : [0,T] \to \left\{ K \subset \bar{\Omega} : K \text{ is } \mathcal{H}^{N-1} \text{ measurable}, \mathcal{H}^{N-1}(K) < \infty \right\}$ is such that:
   (a) $C(0) \overset{\sim}{=} C_0$, for given C_0
   (b) C nondecreasing: $\forall \tau < t$, $C(\tau) \overset{\sim}{\subset} C(t)$
   (c) $\forall t \in [0,T]$, $S(u(t)) \overset{\sim}{\subset} C(t)$
   (d) There exists a function $F : [0,T] \to 2^\Omega$ and a family of functions $v(\cdot,t) : F(t) \to \mathbb{R}$ such that
$$\int_0^T \dot{\varphi}(t) \int_{C(t)} f(x)\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}t = -\int_0^T \varphi(t) \int_{F(t)} f(x)v(x,t)\mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t$$
   $\forall \varphi \in C_0^1([0,T])$, $\forall f \in C_0(\Omega')$.

Definition 6.2. Define the space $\mathcal{T}^*$ to be the set of all pairs (u, C) that satisfy the properties of $\mathcal{T}$ except for property 2d.

Remark 6.3. Note that an alternative to 2b in Definition 6.1 is that v in 2d satisfies $v \geqq 0$. A similar characterization is possible for $q \in \mathcal{T}^*$, requiring the weak derivative of $\mathcal{H}^{N-1} \lfloor C(t)$ to be nonnegative.

Definition 6.4. Define the *rate independent envelope* of ψ, $\bar{\psi} : [0,\infty) \to [0,\infty)$ by

$$\bar{\psi}(x) := \sup_{\substack{\phi \leqq \psi \\ \phi \text{ linear}}} \phi(x).$$

And, setting

$$\mathscr{C} := \inf_{s\in(0,\infty)} \frac{\psi(s)}{s},$$

we have for $s \in [0,\infty)$

$$\bar{\psi}(s) = \mathscr{C}s.$$

6.2. Convergence in $\mathcal{T}^*$

6.2.1. Sketch of compactness argument An important feature of the choice of convergence is that minimizing sequences of (45) are compact. To motivate our choice of convergence, we will briefly sketch the compactness argument for energies of this form. Let $\mathscr{D}$ be a countable, dense subset of $[0,T]$, and suppose that ψ has this property: there exists a constant $\mathcal{K}_1 > 0$ such that, for $s \in [0,\infty)$, ψ satisfies

$$\psi(s) \geqq \mathcal{K}_1 s. \tag{46}$$

The, let $\{q_i = (u_i, C_i)\}_{i=1}^{\infty} \subset \mathcal{T}$ be a minimizing sequence of I_ϵ. This implies that the sequence has bounded energy, that is, there exists $\mathcal{K}_2 > 0$ such that

$$\sup_i I_\epsilon[q_i] < \mathcal{K}_2. \tag{47}$$

We now show that there is a $q = (u, C) \in \mathcal{T}^*$ such that up to a subsequence

$$u_i(\cdot,t) \overset{SBV}{\to} u(\cdot,t)$$

for all t in the countable dense $\mathscr{D} \subset [0,T]$. To see this, we suppose that the minimizing sequence $\{q_i\}_{i=1}^{\infty}$ has the property that for all $i \in \mathbb{N}$ and each $t \in [0,T]$

$$u_i(\cdot,t) \in \operatorname{argmin}\left\{\int_\Omega W(\nabla z)\mathrm{d}x : z \in SBV(\Omega),\ S(z) \,\tilde{\subset}\, C_i(t),\ z = g \text{ in } \Omega'\backslash\Omega\right\},$$

since this can only reduce $I_\epsilon[q_i]$. Then, by the growth bounds on W, $\sup_{i\in\mathbb{N}} \|\nabla u_i(\cdot,t)\|_{L^p(\Omega)}$ is bounded uniformly for $t \in [0,T]$, where $p > 1$. We also assume that our minimizing sequence is chosen so that

$$\sup_{i\in\mathbb{N}} \|u_i(\cdot,t)\|_{L^\infty(\Omega')} \leqq \|g\|_{L^\infty(\Omega')},$$

which, by a truncation argument, can only lower the bulk energy. Now, by (47), we have

$$\int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \int_{F_i(t)} \psi(v_i) \mathrm{d}\mathcal{H}^{N-2} \, \mathrm{d}t < \mathcal{K}_2,$$

which combined with (46) and property 2d of the definition of $\mathcal{T}$ means that there is a $\mathcal{K}_3 > 0$ such that

$$\begin{aligned} \mathcal{H}^{N-1}(C_i(T)) &= \int_0^T \int_{F_i(t)} v_i(x,t) \mathrm{d}\mathcal{H}^{N-2}(x) \mathrm{d}t \\ &< \mathrm{e}^{-\frac{T}{\epsilon}} \mathcal{K}_3. \end{aligned} \tag{48}$$

Then, by the compactness of the space $SBV(\Omega')$ (Theorems 4.7 and 4.8 of [2]), for each $t \in \mathscr{D}$ there is an SBV function u_t such that, up to a subsequence that is not relabeled,

$$u_i(\cdot, t) \overset{SBV}{\rightarrow} u_t.$$

For $t \in \mathscr{D}$, define $u(\cdot, t) := u_t$, and since $\mathscr{D}$ is countable, we apply a diagonal argument to show that up to a subsequence,

$$u_i(\cdot, t) \overset{SBV}{\rightarrow} u(\cdot, t) \tag{49}$$

for all $t \in \mathscr{D}$. Define, for $t \in \mathscr{D}$,

$$C(t) := \bigcup_{\substack{\tau \in \mathscr{D} \\ \tau \leqq t}} S(u(\cdot, \tau)).$$

Then, one would define (u, C) suitably on $[0, T] \backslash \mathscr{D}$, so that $q = (u, C) \in \mathcal{T}^*$. Depending on the specific properties of W and ψ, this convergence can often be stronger. The proof of Theorem 6.13 does not depend on the strength of this convergence, thus we will use a convergence such that (49) holds on the minimal set necessary to build the limiting crack set, giving lower semicontinuity of the energy.

6.2.2. Minimal crack trajectories To define the convergence, we associate to each $q = (u, C) \in \mathcal{T}^*$ the *minimal crack trajectory*, C^*, by the following procedure. For each $t \in [0, T]$ set

$$\mathcal{C}_t := \left\{ K \subset \bar{\Omega} : K \text{ is } \mathcal{H}^{N-1} \text{ measurable, } S(u(\tau)) \cup C_0 \tilde{\subset} K \text{ for all } \tau \leqq t \right\}, \tag{50}$$

and note that

$$\inf_{K \in \mathcal{C}_t} \mathcal{H}^{N-1}(K) \leqq \mathcal{H}^{N-1}(C(T)) < \infty.$$

For each $t \in [0, T]$ take a sequence $\{C_n^t\}_{n=1}^{\infty} \subset \mathcal{C}_t$ such that

$$\mathcal{H}^{N-1}(C_n^t) \rightarrow \inf_{K \in \mathcal{C}_t} \mathcal{H}^{N-1}(K). \tag{51}$$

Define, for $t \in [0, T]$,

$$C^*(t) := \bigcap_{n \in \mathbb{N}} C_n^t. \tag{52}$$

Since for each $t \in [0, T]$, $C_n^t \in \mathcal{C}_t$ for every $n \in \mathbb{N}$, then

$$S(u(\tau)) \cup C_0 \tilde{\subset} C^*(t) \quad \text{for all } \tau \leqq t \tag{53}$$

and since $C^*(t)$ is $\mathcal{H}^{N-1}$ measurable then $C^*(t) \in \mathcal{C}_t$, which by (51) and (52) gives

$$\mathcal{H}^{N-1}(C^*(t)) = \inf_{K \in \mathcal{C}_t} \mathcal{H}^{N-1}(K) \tag{54}$$

for all $t \in [0, T]$. Note $C^*(0) \cong C_0$ and that the map

$$t \mapsto \mathcal{H}^{N-1}(C^*(t))$$

is bounded and monotone, and so is in $BV([0, T])$.

6.2.3. Convergence definition

Definition 6.5. For $q = (u, C) \in \mathcal{T}^*$, with associated C^*, a countable set $\mathscr{D}$ generates q if and only if for every $t \in [0, T]$

$$C^*(t) \cong C_0 \cup \bigcup_{\substack{\tau \leqq t \\ \tau \in \mathscr{D}}} S(u(\tau)).$$

Lemma 6.6. *For any $q = (u, C) \in \mathcal{T}^*$, there exists a countable dense set that generates q.*

Proof. Since the map

$$t \mapsto \mathcal{H}^{N-1}(C^*(t)) \tag{55}$$

is monotone it can only have jump discontinuities, and further these jumps can only occur on a countable subset of $[0, T]$. Choose a countable dense $\mathscr{D}^* \subset [0, T]$ that contains all of the times where the map in (55) has a jump discontinuity. Define, for $t \in [0, T]$ and any countable dense $\mathscr{D} \subset [0, T]$,

$$C(\mathscr{D}, t) := C_0 \cup \bigcup_{\substack{\tau \leqq t \\ \tau \in \mathscr{D}}} S(u(\tau)).$$

Then, for each $t \in \mathscr{D}^*$ take a sequence of countable dense subsets $\{\mathscr{D}_n^t\}_{n=1}^\infty$ such that

$$\mathcal{H}^{N-1}\left(C(\mathscr{D}_n^t, t)\right) \to \sup_{\mathscr{D}'} \mathcal{H}^{N-1}\left(C(\mathscr{D}', t)\right) < \infty.$$

Now, set

$$\mathscr{D}_t := \bigcup_{n\in\mathbb{N}} \mathscr{D}_n^t.$$

Since $\mathscr{D}_t$ is countable and dense then

$$\mathcal{H}^{N-1}\left(C(\mathscr{D}_t, t)\right) = \sup_{\mathscr{D}'} \mathcal{H}^{N-1}\left(C(\mathscr{D}', t)\right).$$

Then, set

$$\mathscr{D} := \bigcup_{t\in\mathscr{D}^*} \mathscr{D}_t,$$

and so at each $t \in \mathscr{D}^*$, we have

$$\mathcal{H}^{N-1}\left(C(\mathscr{D}, t)\right) = \sup_{\mathscr{D}'} \mathcal{H}^{N-1}\left(C(\mathscr{D}', t)\right). \tag{56}$$

$\mathscr{D}$ is a countable dense subset of $[0, T]$, and we will now show that it generates q. First, let $t \in \mathscr{D}^*$. From (53) we have

$$C(\mathscr{D}, t) \mathrel{\tilde{\subset}} C^*(t). \tag{57}$$

For any $t_0 \leqq t$,

$$\mathcal{H}^{N-1}\left(S(u(t_0))\backslash C(\mathscr{D}, t)\right) = 0,$$

since otherwise the countable dense subset $\mathscr{D} \cup \{t_0\}$ would contradict (56). Then since $C(\mathscr{D}, t)$ is $\mathcal{H}^{N-1}$ measurable, it is in $\mathcal{C}_t$ and by (54)

$$\mathcal{H}^{N-1}(C^*(t)) \leqq \mathcal{H}^{N-1}(C(\mathscr{D}, t)).$$

Combining with (57), we have for $t \in \mathscr{D}^*$

$$C^*(t) \cong C(\mathscr{D}, t). \tag{58}$$

Now take $t \in [0, T]\backslash\mathscr{D}^*$. Choose an increasing sequence $\{t_k\}_{k=1}^{\infty} \subset \mathscr{D}^*$ such that $t_k \to t$. Since

$$\begin{aligned} \bigcup_{k\in\mathbb{N}} C^*(t_k) &\cong \bigcup_{k\in\mathbb{N}} C(\mathscr{D}, t_k) \\ &\cong \bigcup_{\substack{\tau<t\\ \tau\in\mathscr{D}}} S(u(\tau)), \end{aligned} \tag{59}$$

then by (53)

$$\bigcup_{k\in\mathbb{N}} C^*(t_k) \mathrel{\tilde{\subset}} C(\mathscr{D}, t) \mathrel{\tilde{\subset}} C^*(t). \tag{60}$$

Therefore

$$\mathcal{H}^{N-1}\left(\bigcup_{k\in\mathbb{N}} C^*(t_k)\right) \leqq \mathcal{H}^{N-1}(C(\mathscr{D},t)) \leqq \mathcal{H}^{N-1}(C^*(t)). \tag{61}$$

By (59) the sequence $\{C^*(t_k)\}_{k=1}^{\infty}$ is nondecreasing and so by choice of the set $\mathscr{D}^*$

$$\begin{aligned}\mathcal{H}^{N-1}\left(\bigcup_{k\in\mathbb{N}} C^*(t_k)\right) &= \lim_{k\to\infty} \mathcal{H}^{N-1}\left(C^*(t_k)\right)\\ &= \mathcal{H}^{N-1}(C^*(t)).\end{aligned}$$

Combining this with (60) and (61) gives

$$C^*(t) \stackrel{\sim}{=} C(\mathscr{D},t).$$

Therefore the set $\mathscr{D}$ generates q. □

Definition 6.7. We will say that $q_i \to q$ (with $\{q_i\}_{i=1}^{\infty} \subset \mathcal{T}^*$, $q \in \mathcal{T}^*$) if and only if

$$u_i(\cdot,t) \overset{SBV}{\to} u(\cdot,t) \quad \text{for all } t \in \mathscr{D} \tag{62}$$

for some countable dense subset $\mathscr{D}$ that generates q.

Remark 6.8. Notice that if a sequence $\{q_i\}_{i=1}^{\infty}$ converges in $\mathcal{T}^*$ the limit is not unique since the limiting C is not uniquely specified.

6.3. Relaxation theorem

The goal of this section is to find a representation for I_ϵ^*, the relaxation of

$$I_\epsilon := \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \int_{F(t)} \psi(v)\mathrm{d}\mathcal{H}^{N-2}\mathrm{d}t$$

with the convergence in (62), that is, for $q \in \mathcal{T}^*$

$$I_\epsilon^*[q] := \inf_{\substack{q_i\in\mathcal{T}\\ q_i\to q}} \left\{\liminf_{i\to\infty} I_\epsilon[q_i]\right\}. \tag{63}$$

Lemma 6.9. *The map*

$$q \mapsto \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}}\mathrm{d}\mu(t), \tag{64}$$

where $q = (u, C) \in \mathcal{T}^*$ with associated C^* and μ is the weak derivative of $t \to \mathcal{H}^{N-1}(C^*(t))$, is lower semicontinuous with the convergence (62) in $\mathcal{T}^*$, that is, whenever $\{q_i\}_{i=1}^{\infty} \subset \mathcal{T}^*$ and $q_i \to q$ in $\mathcal{T}^*$, then

$$\int_0^T \mathrm{e}^{-\frac{t}{\epsilon}}\mathrm{d}\mu(t) \leqq \liminf_{i\to\infty} \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}}\mathrm{d}\mu_i(t).$$

Proof. Let $\mathscr{D}$ generate q and for each $t \in \mathscr{D}$

$$u_i(\cdot, t) \overset{SBV}{\to} u(\cdot, t).$$

This implies, again for each $t \in \mathscr{D}$

$$\mathcal{H}^{N-1}(S(u(\cdot, t)) \leqq \liminf_{i\to\infty} \mathcal{H}^{N-1}(S(u_i(\cdot, t))).$$

By Lemma 3.1 in [6], we then have for all $t \in [0, T]$

$$\mathcal{H}^{N-1}(C^*(t)) = \mathcal{H}^{N-1}\left(\bigcup_{\substack{\tau\leqq t\\ \tau\in\mathscr{D}}} S(u(\cdot, t))\right) \leqq \liminf_{i\to\infty} \mathcal{H}^{N-1}\left(\bigcup_{\substack{\tau\leqq t\\ \tau\in\mathscr{D}}} S(u_i(\cdot, t))\right).$$

Denoting the minimal crack trajectories associated to q_i by C_i^*, we then have

$$\mathcal{H}^{N-1}(C^*(t)) \leqq \liminf_{i\to\infty} \mathcal{H}^{N-1}(C_i^*(t)) \tag{65}$$

for any $t \in [0, T]$. Applying an integration by parts to the map in (64) gives

$$\int_0^T e^{-\frac{t}{\epsilon}} d\mu(t) = \epsilon \int_0^T e^{-\frac{t}{\epsilon}} \mathcal{H}^{N-1}(C^*(t))dt + e^{-\frac{T}{\epsilon}} \mathcal{H}^{N-1}(C^*(T)) - \mathcal{H}^{N-1}(C_0). \tag{66}$$

Combine (65) and (66) with Fatou's Lemma and the lemma is proved. $\square$

Lemma 6.10. *Suppose $\{u_i\}_{i=1}^{\infty} \subset SBV_p(\Omega)$, $p > 1$, such that $\mathcal{H}^{N-1}\left(\bigcup_{i=1}^{\infty} S(u_i)\right) < \mathcal{C}$, for some constant $\mathcal{C}$. Then, $\exists v \in SBV(\Omega)$ such that*

$$\bigcup_{i=1}^{\infty} S(u_i) \cong S(v).$$

Proof. First, we assume that for each $i \in \mathbb{N}$, $u_i \in L^{\infty}(\Omega)$, since for any $w \in SBV(\Omega)$, $\arctan(w) \in SBV(\Omega) \cap L^{\infty}(\Omega)$ and

$$S(\arctan(w)) = S(w).$$

The plan is to define a sequence $\{v_i\}_{i=1}^{\infty}$ by

$$v_i := \sum_{j=1}^{i} r_j u_j,$$

where the constants $\{r_j\}_{j=1}^{\infty}$ will be chosen so that three properties hold. First, $\{v_i\}_{i=1}^{\infty}$ will converge in SBV to some v. Also, we will have that for any $i \in \mathbb{N}$,

$$\bigcup_{j=1}^{i} S(u_j) \cong S(v_i).$$

Finally, we will have that for every $i \in \mathbb{N}$ there is a constant $\eta_i > 0$ such that, for all $k > i$ and $x \in S(v_k)$ (except on a set whose $\mathcal{H}^{N-1}$ measure is less than $1/i$),

$$|[v_k](x)| > \eta_i > 0,$$

which means that the jump sets of the $\{v_i\}_{i=1}^{\infty}$ do not disappear in the limit. We begin by setting

$$r_1 := \frac{1}{2 \max\left\{1, \|\nabla u_1\|_{L^p(\Omega)}\right\} \max\left\{1, \|u_1\|_{L^\infty(\Omega)}\right\}}$$

and then let $v_1 := r_1 u_1$. As in [6] (see Lemma 3.1), given $\{v_j\}_{j=1}^{i-1} \subset SBV(\Omega)$, $\mathcal{H}^{N-1}(S(v_j)) < \mathcal{C}\ \forall j \in \mathbb{N}$, set

$$A^{i-1}(\xi) := \{x \in S(v_{i-1}) : [v_{i-1}](x) + \xi[u_i](x) = 0\},$$

where, for any $z \in SBV(\Omega)$ and $x \in S(z)$, $[z](x)$ denotes the jump in the trace from either side of $S(z)$ at x, that is, $[z](x) := z^+(x) - z^-(x)$. Note that since the sets $A^{i-1}(\xi)$, $\xi \in \mathbb{R}$, are disjoint and measurable, $\mathcal{H}^{N-1}(A^{i-1}(\xi)) = 0$ except possibly for countably many values of ξ. Choose $\delta_{i-1} \in (0, \delta_{i-2})$ (taking $\delta_0 := 1$) such that

$$\mathcal{H}^{N-1}\left(\{x \in S(v_{i-1}) : |[v_{i-1}](x)| \leqq \delta_{i-1}\}\right) < \frac{1}{i-1}.$$

Choose $r_i \in (0, r_{i-1})$, such that

1. $r_i < \dfrac{\delta_{i-1}}{2^i \max\left\{1, \|\nabla u_i\|_{L^p(\Omega)}\right\} \max\left\{1, \|u_i\|_{L^\infty(\Omega)}\right\}}$ and
2. $\mathcal{H}^{N-1}(A^{i-1}(r_i)) = 0.$

Now set

$$v_i := v_{i-1} + r_i u_i = \sum_{j=1}^{i} r_j u_j.$$

By the choice of $\{r_i\}_{i=1}^{\infty}$, specifically property 2, we have that

$$S(u_j) \,\tilde{\subset}\, S(v_k), \quad \forall k \geqq j. \tag{67}$$

Also by the choice of the $\{r_i\}_{i=1}^{\infty}$ (property 1), we have that

$$\begin{aligned} \|\nabla v_i\|_{L^p(\Omega)} &\leqq \sum_{j=1}^{i} \frac{1}{2^j \max\left\{1, \left\|\nabla u_j\right\|_{L^p(\Omega)}\right\}} \left\|\nabla u_j\right\|_{L^p(\Omega)} \\ &\leqq 1, \end{aligned} \tag{68}$$

and

$$
\begin{aligned}
\|v_i\|_{L^\infty(\Omega)} &\leqq \sum_{j=1}^{i} \frac{1}{2^j \max\left\{1, \|u_j\|_{L^\infty(\Omega)}\right\}} \|u_j\|_{L^\infty(\Omega)} \\
&\leqq 1. \qquad (69)
\end{aligned}
$$

These two estimates, the uniform bound on $\mathcal{H}^{N-1}(S(v_i))$, and the compactness of the space $SBV(\Omega)$ (Theorems 4.7 and 4.8 of [2]) imply that there exists $v \in SBV(\Omega)$ such that, up to a subsequence,

$$
[v_i]\mathcal{H}^{N-1}\lfloor S(v_i) \overset{*}{\rightharpoonup} [v]\mathcal{H}^{N-1}\lfloor S(v). \qquad (70)
$$

Further, by the calculation in (69), the sequence $\{v_i\}_{i=1}^\infty$ is a Cauchy sequence in L^∞, and so converges to some $v \in L^\infty$. The uniqueness of that limit implies that the convergence in (70) holds without dropping to a subsequence. Now, by (67) we can show that

$$
\bigcup_{i=1}^{\infty} S(u_i) \mathrel{\tilde{\subset}} S(v), \qquad (71)
$$

by proving that

$$
\bigcup_{i=1}^{\infty} S(v_i) \mathrel{\tilde{\subset}} S(v). \qquad (72)
$$

So, fix $i \in \mathbb{N}$, and let $\gamma > 0$. Choose $M \in \mathbb{N}$ large enough so that $M > i$ and $1/M < \gamma$. For $k > M$,

$$
S(v_i) \mathrel{\tilde{\subset}} S(v_k), \qquad (73)
$$

and setting

$$
B_k := \left\{x \in S(v_k) : |[v_k](x)| \leqq \delta_k\right\}
$$

we have, by the choice of the sequence $\{\delta_k\}_{k=1}^\infty$,

$$
\mathcal{H}^{N-1}(B_k) < \gamma. \qquad (74)
$$

This implies that, for $x \in S(v_k)\backslash B_k$,

$$
\begin{aligned}
\left|\sum_{i=k+1}^{\infty} r_i[u_i](x)\right| &\leqq \left|\sum_{i=k+1}^{\infty} \frac{\delta_{i-1}}{2^i \max\left\{1, \|u_i\|_{L^\infty(\Omega)}\right\}} 2\|u_i\|_{L^\infty(\Omega)}\right| \\
&\leqq \left|\sum_{i=k+1}^{\infty} \frac{\delta_{i-1}}{2^{i-1}}\right| \\
&\leqq \left|\delta_k \sum_{i=k}^{\infty} \frac{1}{2^i}\right| \\
&< |[v_k](x)|.
\end{aligned}
$$

Therefore

$$S(v_k) \mathrel{\tilde{\subset}} (B_k \cup S(v)),$$

by (74) we have

$$\mathcal{H}^{N-1}(S(v_k) \backslash S(v)) < \gamma,$$

and so (73) implies

$$\mathcal{H}^{N-1}(S(v_i) \backslash S(v)) < \gamma.$$

Since γ was arbitrary, we have

$$S(v_i) \mathrel{\tilde{\subset}} S(v),$$

and since i was arbitrary, we have (72) and we have proved (71). The inclusion

$$S(v) \mathrel{\tilde{\subset}} \bigcup_{i=1}^{\infty} S(u_i)$$

follows from (70). □

Remark 6.11. Note that the rate independent envelope gives the optimal dissipation and front speed. For any $q = (u, C) \in \mathcal{T}$ and any $t_1 < t_2$, we have

$$\begin{aligned}
\int_{t_1}^{t_2} \int_{F(t)} \psi(v(x,t)) \mathrm{d}\mathcal{H}^{N-2}(x)\, \mathrm{d}t &\geqq \int_{t_1}^{t_2} \int_{F(t)} \bar{\psi}(v(x,t)) \mathrm{d}\mathcal{H}^{N-2}(x) \mathrm{d}t \\
&= \mathscr{C} \int_{t_1}^{t_2} \int_{F(t)} v(x,t) \mathrm{d}\mathcal{H}^{N-2}(x) \mathrm{d}t \\
&= \mathscr{C} \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{H}^{N-1}(C(t)) \mathrm{d}t \\
&= \mathscr{C} \mathcal{H}^{N-1}(C(t_2) \backslash C(t_1)).
\end{aligned}$$

Also, by the continuity of ψ, there is a sequence of front speeds $\{v_i\}_{i=1}^{\infty}$ such that

$$\frac{\psi(v_i)}{v_i} \to \mathscr{C}.$$

We now show that this optimal front speed, and with it the optimal dissipation, can be achieved by using the right front geometry.

Theorem 6.12. *Let* $[a, b] \subset [0, T]$ *and* $\Gamma \subset \bar{\Omega}$, $\mathcal{H}^{N-1}(\Gamma) < \infty$, *such that* $\Gamma \mathrel{\tilde{=}} S(w)$ *for some* $w \in SBV(\Omega)$. *Then, for any* $\delta > 0$, *there is pair* (C_δ, F_δ), *defined for* $t \in [a, b]$, $C_\delta(b) \backslash C_\delta(a) \mathrel{\tilde{=}} \Gamma$, *the pair satisfies the properties of part 2 of Definition 6.1 (in particular the front representation formula with front speed that we denote* v_δ*), and*

$$\int_a^b \int_{F_\delta(t)} \psi\left(v_\delta(x,t)\right) \mathrm{d}\mathcal{H}^{N-2}(x) \mathrm{d}t < (1+\delta) \mathscr{C} \mathcal{H}^{N-1}(\Gamma). \tag{75}$$

Proof. The plan is to cover Γ with a countable collection of cubes so that in each cube Γ is close to a hyperplane through the center of the cube. We partition $[a, b]$ into a countable family of subintervals. In each cube we will construct (C_δ, F_δ) during one of the time subintervals by taking $N-1$ dimensional slices of Γ that move at a speed which gives the optimal front speed, according to Remark 6.11. In each cube we will miss subsets of Γ of small $\mathcal{H}^{N-1}$ measure, for which we later repeat the above process, and in the end we will miss only a set of $\mathcal{H}^{N-1}$ measure zero.

Let $A_1 = \Gamma$; in what follows we will inductively define $\{A_k\}_{k=2}^{\infty}$, $A_k \subset A_{k-1}$ for all $k \in \mathbb{N}$.

Part I First we divide $[a, b]$. Let $\{I_k\}_{k=1}^{\infty}$, $I_k \subset [a, b]$ $\forall k \in \mathbb{N}$, be a countable, disjoint collection of intervals such that each I_k is nonempty and so that

$$\mathcal{L}^1\left([a, b] \triangle \bigcup_{k=1}^{\infty} I_k\right) = 0.$$

Then, for each I_k, let $\{Y_\ell^k\}_{\ell=1}^{\infty}$, $Y_\ell^k \subset I_k$ $\forall \ell \in \mathbb{N}$, be a countable disjoint collection of intervals, each nonempty, such that

$$\mathcal{L}^1\left(I_k \triangle \bigcup_{\ell=1}^{\infty} Y_\ell^k\right) = 0.$$

So, we have that:

$$\mathcal{L}^1\left([a, b] \triangle \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} Y_\ell^k\right) = 0.$$

Part II Suppose we have defined $\{A_j\}_{j=1}^{k}$, with $A_j \subset A_{j-1} \subset \Gamma$ for $j = 1, \ldots, k$. As outlined above, we will now cover A_k with a suitable family of cubes in order to define the crack trajectory and crack front. As in the proof of Theorem 2.1 in [6], let $\mathcal{D}$ be a countable dense subset of $\mathbb{R}$ such that for each $\xi \in \mathcal{D}$, E_ξ^w is a set of finite perimeter. Then

$$S(w) \mathrel{\tilde{\subset}} \bigcup_{\xi \in \mathcal{D}} \partial^* E_\xi^w.$$

Let $\eta > 0$. From now on, if $x_0 \in \partial^* E$ for some specified set of finite perimeter E, assume that any cube $Q(x_0, r)$ is oriented so that $\nu_E(x_0)$ is normal to one of the faces of the cube. From [6] (see the derivation of equation (2.1) in the proof of Theorem 2.1), we know that for all $\xi \in \mathcal{D}$, and $\mathcal{H}^{N-1}$-almost every $x \in A_k \cap \partial^* E_\xi^w$,

$$\lim_{r \downarrow 0} \frac{\mathcal{H}^{N-1}(Q(x, r) \cap A_k \cap \partial^* E_\xi^w)}{(2r)^{N-1}} = 1. \tag{76}$$

We have for $x \in \partial^* E_\xi^w$ (see Remark 3.55 in [2])

$$\lim_{r \downarrow 0} \fint_{Q(x,r)} |\nu_{E_\xi^w}(y) - \nu_{E_\xi^w}(x)| \mathrm{d}|D\chi_{E_\xi^w}|(y) = 0.$$

This implies that

$$\lim_{r\downarrow 0}\frac{|D\chi_{E_\xi^w}|\Big(\{y\in Q(x,r):|\nu_{E_\xi^w}(y)-\nu_{E_\xi^w}(x)|\geqq\eta\}\Big)}{|D\chi_{E_\xi^w}|(Q(x,r))}=0$$

and so

$$\lim_{r\downarrow 0}\frac{|D\chi_{E_\xi^w}|\Big(\{y\in Q(x,r):|\nu_{E_\xi^w}(y)-\nu_{E_\xi^w}(x)|<\eta\}\Big)}{|D\chi_{E_\xi^w}|(Q(x,r))}=1$$

for $x\in\partial^* E_\xi^w$. Combining this with Corollary 1 of Section 5.7 in [5], we then have that, again for $x\in\partial^* E_\xi^w$,

$$\lim_{r\downarrow 0}\frac{|D\chi_{E_\xi^w}|\Big(\{y\in Q(x,r):|\nu_{E_\xi^w}(y)-\nu_{E_\xi^w}(x)|<\eta\}\Big)}{(2r)^{N-1}}=1.$$

And, since by Theorem 2 in Section 5.7 of [5], we have $|D\chi_{E_\xi^w}|=\mathcal{H}^{N-1}\lfloor\partial^* E_\xi^w$, we have that for $x\in\partial^* E_\xi^w$

$$\lim_{r\downarrow 0}\frac{\mathcal{H}^{N-1}\Big(Q(x,r)\cap\{y\in\partial^* E_\xi^w:|\nu_{E_\xi^w}(y)-\nu_{E_\xi^w}(x)|<\eta\}\Big)}{(2r)^{N-1}}=1. \tag{77}$$

Combining (76) and (77), we know that for all $\xi\in\mathcal{D}$ and $\mathcal{H}^{N-1}$-almost every $x\in A_k\cap\partial^* E_\xi^w$,

$$\lim_{r\downarrow 0}\frac{\mathcal{H}^{N-1}\Big(Q(x,r)\cap A_k\cap\{y\in\partial^* E_\xi^w:|\nu_{E_\xi^w}(y)-\nu_{E_\xi^w}(x)|<\eta\}\Big)}{(2r)^{N-1}}=1. \tag{78}$$

Now, since $\mathcal{D}$ is countable, we also have that (78) holds for $\mathcal{H}^{N-1}$-almost every $x\in A_k$ and all $\xi\in\mathcal{D}$ such that $x\in\partial^* E_\xi^w$.

For $\mathcal{H}^{N-1}$-almost every $x\in A_k$, we choose $\xi(x)$ such that for the set $E_x:=E_{\xi(x)}^w$ we have $x\in\partial^* E_x$. We use (78) to finely cover (up to a set of $\mathcal{H}^{N-1}$ measure zero) the set A_k with the family $\mathcal{G}$ of all cubes $Q(x,r)$, $x\in A_k$, and r small enough so that $Q(x,r)\subset\Omega'$ and the following properties hold:

1. $\left(1-\frac{\eta}{k}\right)(2r)^{N-1}<\mathcal{H}^{N-1}(Q(x,r)\cap A_k\cap\{y\in\partial^* E_x:|\nu_{E_x}(y)-\nu_{E_x}(x)|<\eta\})<\left(1+\frac{\eta}{k}\right)(2r)^{N-1}$
2. $\left(1-\frac{\eta}{k}\right)(2r)^{N-1}<\mathcal{H}^{N-1}(Q(x,r)\cap A_k)<\left(1+\frac{\eta}{k}\right)(2r)^{N-1}$.

Now, applying Besicovitch's covering theorem (specifically Corollary 1 of Section 1.5 in [5]) using the Radon measure $\mathcal{H}^{N-1}\lfloor A_k$, we get a countable disjoint collection of cubes $\{Q_\ell^k\}_{\ell=1}^\infty\subset\mathcal{G}$, such that

$$\mathcal{H}^{N-1}\left(A_k\backslash\bigcup_{\ell=1}^\infty Q_\ell^k\right)=0.$$

In each cube Q^k_ℓ, we will build up the set $A_k \cap Q^k_\ell$ in the time interval Y^k_ℓ, in a way that has a front representation, and uses the optimal front speed as calculated in Part I.

Part III Fix such a pair (Q^k_ℓ, Y^k_ℓ), and we will employ the simpler notation $Y^k_\ell = [t_1, t_2]$, $\Delta t := t_2 - t_1$ and $Q^k_\ell = Q(x, r)$. Also, we assume a coordinate system so that

$$Q^k_\ell = \prod_{i=1}^{N} [0, 2r]$$

and $\nu_{E_x}(x) = e_1$. Define

$$G^k_\ell := Q^k_\ell \cap A_k \cap \left\{ y \in \partial^* E_x : \nu^1_{E_x}(y) > 1 - \eta \right\}.$$

Note that by properties 1 and 2 of the choice of cubes, we have

$$\mathcal{H}^{N-1}\left(Q^k_\ell \cap \left(A_k \backslash G^k_\ell\right)\right) < \frac{2}{k}\eta(2r)^{N-1}.$$

The plan is to define a front by taking $N-1$ dimensional slices of the set G^k_ℓ. With this in mind, define the "slicing function" σ, which maps pairs $(t, A) \in \mathbb{R} \times \mathbb{R}^N$ to subsets of $\mathbb{R}^{N-1}$ by

$$\sigma(t, A) := \left\{ z \in \mathbb{R}^{N-1} : (z^1, \ldots, z^{N-1}, t) \in A \right\}.$$

Also, define the family of imbeddings of $\mathbb{R}^{N-1}$ into $\mathbb{R}^N$ by setting for $t \in \mathbb{R}$ and $\tilde{A} \subset \mathbb{R}^{N-1}$:

$$\phi_t(\tilde{A}) := \left\{ y \in \mathbb{R}^N : y = (z^1, \ldots, z^{N-1}, t) \text{ for some } z \in \tilde{A} \right\}.$$

Set

$$S_t := \sigma\left(t, Q^k_\ell \cap E_x\right).$$

Claim. For $\mathcal{L}^1$-almost every $t \in [0, 2r]$, S_t is a set of finite perimeter in $\mathbb{R}^{N-1}$. (79)

Proof of Claim. By Theorem 2 in Section 5.10 of [5], we know that $f \in BV_{\text{loc}}(\mathbb{R}^N)$ if and only if

$$\int_K (\text{ess } V_a^b f_k)(x') \mathrm{d}\mathcal{L}^{N-1}(x') < \infty, \tag{80}$$

for each $k = 1, \ldots, N$, $a < b$, and compact set $K \subset \mathbb{R}^{N-1}$, with $x' = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_N) \in \mathbb{R}^{N-1}$ and

$$f_k(x', t) := f(\ldots, x_{k-1}, t, x_{k+1}, \ldots).$$

Let

$$K^* := \left(\prod_{i=1}^{N-1}[0, 2r]\right) \subset \mathbb{R}^{N-1}.$$

For any $y \in K^*$, define the function $(\chi_{E_x})_y : (0, 2r) \to \{0, 1\}$ by

$$s \mapsto (\chi_{E_x})_y(s) := \chi_{E_x \cap Q_\ell^k}(s, y).$$

Also, define the function $SV : K^* \to \mathbb{R}$ by

$$y \mapsto SV(y) := \text{ ess } V_0^{2r}(\chi_{E_x})_y.$$

Since $\chi_{E_x} \in BV(\Omega)$, then applying (80), using K^* as our compact set, gives

$$\int_{K^*} SV(y)\mathrm{d}\mathcal{L}^{N-1}(y) < \infty. \tag{81}$$

Then, if $N = 2$, we have proven (79), since for any $s, t \in (0, 2r)$, $(\chi_{E_x})_t(s) = \chi_{S_t}(s)$ and so by (81), for $\mathcal{L}^1$-almost every t, χ_{S_t} has finite essential variation. For $N > 2$, let

$$K^{**} := \left(\prod_{i=1}^{N-2}[0, 2r]\right) \subset \mathbb{R}^{N-2}.$$

Then applying Fubini's theorem to (81) we have

$$\int_0^{2r}\int_{K^{**}} SV(y', \xi)\mathrm{d}\mathcal{L}^{N-2}(y')\mathrm{d}\xi < \infty.$$

So, there exists a set $\mathcal{N} \subset [0, 2r]$ such that for $\xi \in [0, 2r]\backslash\mathcal{N}$,

$$\int_{K^{**}} SV(y', \xi)\mathrm{d}\mathcal{L}^{N-2}(y') < \infty.$$

and

$$\mathcal{L}^1([0, 2r]\backslash\mathcal{N}) = 0.$$

For any $t \in [0, 2r]\backslash\mathcal{N}$, and $y' \in K^{**}$, define the function $(\chi_{\sigma_t})_{y'} : (0, 2r) \to \{0, 1\}$ by

$$z \mapsto (\chi_{\sigma_t})_{y'}(z) := \chi_{\sigma(t, E_x \cap Q_\ell^k)}(z, y'),$$

and then define the function $SV_t : K^{**} \to \mathbb{R}$

$$y' \mapsto SV_t(y') := \text{ ess } V_0^{2r}(\chi_{\sigma_t})_{y'}.$$

By definition of σ, we have that for any $t \in [0, 2r]$, $y' \in K^{**}$, and $z \in (0, 2r)$:

$$(\chi_{\sigma_t})_{y'}(z) = (\chi_{E_x})_{(y', t)}(z),$$

and so

$$SV_t(y') = SV(y', t)$$

for all $y' \in K^{**}$, $t \in [0, 2r]$. Therefore, for $t \in [0, 2r]\backslash\mathcal{N}$,

$$\int_{K^{**}} SV_t(y')\mathrm{d}\mathcal{L}^{N-1}(y') < \infty. \tag{82}$$

Applying (82) and the other implication of Theorem 2 in Section 5.10 of [5] to the function $\chi_{\sigma(t, Q_\ell^k \cap E_x)}$ defined on $\mathbb{R}^{N-1}$, gives us that for $\mathcal{L}^1$-almost every $t \in [0, 2r]$, $\chi_{\sigma(t, Q_\ell^k \cap E_x)} \in BV(\mathbb{R}^{N-1})$, which means that the set S_t is a set of finite perimeter in $\mathbb{R}^{N-1}$, which concludes the proof of (79). □

The above claim implies that there exists a set $\mathcal{N} \subset [0, 2r]$ with measure zero such that, for $t \in [0, 2r]\backslash\mathcal{N}$, there exists a vector valued Radon measure on $\mathbb{R}^{N-1}$, denoted

$$[\partial S_t] = \left(|\partial_{e_1} S_t|, \ldots, |\partial_{e_{N-1}} S_t|\right),$$

such that

$$\int_{\sigma(t, Q_\ell^k)} \chi_{S_t}(y)\,\mathrm{div}\,\varphi(y)\mathrm{d}\mathcal{L}^{N-1}(y) = -\int_{\sigma(t, Q_\ell^k)} \varphi(y)\cdot \mathrm{d}[\partial S_t](y)$$

for all $\varphi \in C_0^1(\sigma(t, Q_\ell^k); \mathbb{R}^{N-1})$. And, according to Theorem 2 in Section 5.7 of [5], we have that

$$|\partial S_t| = \mathcal{H}^{N-2}\llcorner\partial^* S_t$$

for $t \notin \mathcal{N}$.

Part IV The goal of this part of the proof is to show how $A_k \cap Q_l^k$ can be built up in a way that satisfies the front representation formula by taking a moving slice of the cube with speed 1. Define, for $t \in [0, 2r]$,

$$\tilde{F}(t) := \begin{cases} \phi_t(\sigma(t, G_\ell^k) \cap \partial^* S_t) & \text{if } \ t \notin \mathcal{N} \\ \emptyset & \text{if } \ t \in \mathcal{N} \end{cases}$$

and

$$\tilde{C}(t) := \left\{ y \in G_\ell^k : y^N \leqq t \right\}.$$

For every $t \in [0, 2r]$, $\tilde{C}(t)$ is the intersection of a $|D\chi_{E_x}|$ measurable set and a Borel set and therefore is $|D\chi_{E_x}|$ measurable. Also, $\tilde{C}(2r) = G_\ell^k$. To show that the pair $(\tilde{C}, \tilde{F})$ satisfies the front representation formula, we will define a family of measures ρ_t, $t \in [0, 2r]$, such that

$$\rho_t(A) = \int_0^t \mathcal{H}^{N-2}(\tilde{F}(\xi) \cap A)\mathrm{d}\xi, \tag{83}$$

for any Borel set $A \subset \mathbb{R}^N$. First, we must ensure that a family of Radon measures can be defined in this manner.

For $j < N$, the measure valued map

$$t \mapsto \begin{cases} |\partial_{e_j} S_t| & \text{if } t \in [0, 2r] \backslash \mathcal{N} \\ 0 & \text{if } t \in \mathcal{N} \end{cases} \tag{84}$$

is $\mathcal{L}^1$-measurable in the sense of Definition 2.25 of [2] by the following adaptation of Lemma 3.106 in [2]. By Proposition 2.6 of [2], we need to verify that for any open set $A \subset Q^k_\ell$, the map $t \mapsto |\partial_{e_j} S_t|(A)$ is $\mathcal{L}^1$-measurable. Taking A to be such a set, we choose a sequence $f_n \to e_j \chi_A$, $f_n \in C^1_0(A; \mathbb{R}^{N-1})$. Then, the functions

$$t \mapsto \Psi_n(t) := \int_{\sigma(t, Q^k_\ell)} \chi_{S_t}(\xi) \operatorname{div} f_n(\xi) \mathrm{d}\mathcal{L}^{N-1}(\xi)$$

are $\mathcal{L}^1$-measurable for all n by Fubini's Theorem. Since for all $n \in \mathbb{N}$

$$\int_{\sigma(t, Q^k_\ell)} \chi_{S_t}(\xi) \operatorname{div} f_n(\xi) \mathrm{d}\mathcal{L}^{N-1}(\xi) = -\int_{\sigma(t, Q^k_\ell)} f_n(\xi) \cdot \mathrm{d}[\partial S_t](\xi),$$

then for $\mathcal{L}^1$-almost every t,

$$-\Psi_n(t) \to |\partial_{e_j} S_t|(A),$$

as $n \to \infty$, and so we satisfy the requirement of Proposition 2.6 of [2], which implies that the map in (84) is $\mathcal{L}^1$-measurable. Further, by Theorem 3.107 in [2], we have for any $j < N$,

$$|D_{e_j} \chi_{E_x}| = \mathcal{L}^1 \lfloor [0, 2r] \otimes |\partial_{e_j} S_t|, \tag{85}$$

where the measure product on the right-hand side is given by Definition 2.27 of [2]:

$$\left(\mathcal{L}^1 \lfloor [0, 2r] \otimes |\partial_{e_j} S_t|\right)(A) := \int_0^{2r} \int_{\sigma(t, Q^k_\ell)} \chi_{\sigma(t,A)}(\xi) \mathrm{d}|\partial_{e_j} S_t|(\xi) \mathrm{d}t,$$

for any $A \subset Q^k_\ell$, A Borel. Since

$$\begin{aligned} |D_{e_j} \chi_{E_x}|(Q^k_\ell) &\leqq |D\chi_{E_x}|(Q^k_\ell) \\ &< \infty, \end{aligned}$$

the measure $\mathcal{L}^1 \lfloor [0, 2r] \otimes |\partial_{e_j} S_t|$ is Radon, again for $j < N$. Next, we turn our attention to the measure-valued map

$$t \mapsto \begin{cases} |\partial S_t| & \text{if } t \in [0, 2r] \backslash \mathcal{N} \\ 0 & \text{if } t \in \mathcal{N}. \end{cases} \tag{86}$$

For any $j < N$, the function ζ_j, defined for $t \in [0, 2r]$ and $x \in \prod_{i=1}^{N-1} [0, 2r]$ (up to a set of $\mathcal{L}^1 \lfloor [0, 2r] \otimes |\partial_{e_j} S_t|$ measure zero)

$$\zeta_j(t, x) := \nu^j_{S_t}(x),$$

is $\mathcal{L}^1\lfloor[0,2r]\otimes|\partial_{e_j}S_t|$-measurable, and so it follows that $(\zeta_j)^2$ is $\mathcal{L}^1\lfloor[0,2r]\otimes|\partial_{e_j}S_t|$-measurable. Proposition 2.26 of [2] implies that for all $j<N$, the map

$$t\mapsto\int_{\sigma(t,Q^k_\ell)}\left(\nu^j_{S_t}\right)^2(x)\mathrm{d}|\partial S_t|(x)$$

is $\mathcal{L}^1\lfloor[0,2r]$ measurable, which implies that the map in (86) is $\mathcal{L}^1\lfloor[0,2r]$ measurable. Also, for any $j<N$,

$$\int_0^{2r}\nu^j_{S_t}(\xi)\mathrm{d}|\partial S_t|(\xi)\mathrm{d}t=\int_0^{2r}|\partial_{e_j}S_t|(\sigma(t,Q^k_\ell))\mathrm{d}t$$
$$<\infty,$$

and so, since for any $t\in[0,2r]\backslash\mathcal{N}$ and $\xi\in\mathbb{R}^{N-1}$, $\sum_{j=1}^{N-1}(\nu^j_{S_t})^2(\xi)=1$, we have

$$\begin{aligned}\int_0^{2r}|\partial S_t|(\sigma(t,Q^k_\ell))\mathrm{d}t&=\int_0^{2r}\int_{\sigma(t,Q^k_\ell)}\sum_{j=1}^{N-1}\left\{(\nu^j_{S_t})^2(\xi)\right\}\mathrm{d}|\partial S_t|(\xi)\mathrm{d}t\\&=\sum_{j=1}^{N-1}\int_0^{2r}\int_{\sigma(t,Q^k_\ell)}(\nu^j_{S_t})^2(\xi)\mathrm{d}|\partial S_t|(\xi)\mathrm{d}t\\&\leqq\sum_{j=1}^{N-1}\int_0^{2r}\int_{\sigma(t,Q^k_\ell)}\nu^j_{S_t}(\xi)\mathrm{d}|\partial S_t|(\xi)\mathrm{d}t\\&<\infty.\end{aligned}$$

Therefore, we define the Radon measure by the measure product

$$\left(\mathcal{L}^1\lfloor[0,2r]\otimes|\partial S_t|\right)(A):=\int_0^{2r}\int_{\sigma(t,Q^k_\ell)}\chi_{\sigma(t,A)}\mathrm{d}|\partial S_t|(\xi)\mathrm{d}t,$$

for all $A\subset Q^k_\ell$, A Borel. Since the set G^k_ℓ is $|D\chi_{E_x}|$ measurable, there exists a Borel set that agrees $|D\chi_{E_x}|$-almost everywhere with G^k_ℓ, and so we assume that G^k_ℓ is Borel. Therefore, we can define the family of Radon measures, $t\in[0,2r]$, by setting for each Borel set $A\subset Q^k_\ell$

$$\begin{aligned}\rho_t(A)&:=\int_A\chi_{G^k_\ell}(y)\mathrm{d}\left(\mathcal{L}^1\lfloor[0,t]\otimes|\partial S_\xi|\right)(y)\\&=\int_0^t\int_{\sigma(\xi,G^k_\ell\cap A)}\mathrm{d}|\partial S_\xi|\mathrm{d}\xi.\end{aligned}$$

Since $|\partial S_t|=\mathcal{H}^{N-2}\lfloor\partial^*S_t$, and by definition of $\tilde{F}$, we can write these measures as

$$\begin{aligned}\rho_t(A)&=\int_0^t\mathcal{H}^{N-2}\left(\sigma(\xi,G^k_\ell\cap A)\cap\partial^*S_\xi\right)\mathrm{d}\xi\\&=\int_0^t\mathcal{H}^{N-2}\left(\phi_t(\sigma(\xi,G^k_\ell\cap A)\cap\partial^*S_\xi)\right)\mathrm{d}\xi\\&=\int_0^t\mathcal{H}^{N-2}\left(\tilde{F}(\xi)\cap A\right)\mathrm{d}\xi\end{aligned}\tag{87}$$

giving (83).
Next, we show that $\forall t \in [0, 2r]$, we have that for any ball $B \subset Q_\ell^k$

$$(1-\eta)|D\chi_{E_x}|(\tilde{C}(t) \cap B) \leqq \rho_t(B) \leqq |D\chi_{E_x}|(\tilde{C}(t) \cap B). \tag{88}$$

We have by choice of the set G_ℓ^k,

$$(1-\eta)|D\chi_{E_x}|(\tilde{C}(t) \cap B) \leqq |D_{e_1}\chi_{E_x}|(\tilde{C}(t) \cap B).$$

Using (85), we have

$$\begin{aligned}
|D_{e_1}\chi_{E_x}|(\tilde{C}(t) \cap B)) &= \int_0^t |\partial_{e_1} S_\xi| \left(\sigma(\xi, \tilde{C}(t)) \cap \sigma(\xi, B)\right) \mathrm{d}\xi \\
&= \int_0^t |\partial_{e_1} S_\xi| \left(\sigma(\xi, G_\ell^k) \cap \sigma(\xi, B)\right) \mathrm{d}\xi \\
&\leqq \int_0^t |\partial S_\xi| \left(\sigma(\xi, G_\ell^k) \cap \sigma(\xi, B)\right) \mathrm{d}\xi \\
&= \int_0^t \mathcal{H}^{N-2} \left(\phi_\xi(\sigma(\xi, G_\ell^k) \cap \partial^* S_\xi \cap B)\right) \mathrm{d}\xi \\
&= \rho_t(B),
\end{aligned}$$

and so (88) is proved. This estimate implies that, $\forall t \in [0, 2r]$,

$$|D\chi_{E_x}|\lfloor\tilde{C}(t) << \rho_t,$$

and that the densities

$$\gamma_t(\xi) := \frac{\mathrm{d}(|D\chi_{E_x}|\lfloor\tilde{C}(t))}{\mathrm{d}\rho_t}(\xi)$$

exist $\forall t \in [0, 2r]$, ρ_t-almost everywhere and satisfy the uniform bounds

$$1 \leqq \gamma_t \leqq \frac{1}{1-\eta}.$$

Therefore, by the generalized Fubini theorem of Definition 2.27 in [2], we have for all $\varphi \in C_0^1([0, 2r])$ and $f \in C_0(Q(x, r))$,

$$\begin{aligned}
\int_0^{2r} \dot{\varphi}(t) \int_{\tilde{C}(t)} f(x) \mathrm{d}\mathcal{H}^{N-1}(x) \mathrm{d}t &= \int_0^{2r} \dot{\varphi}(t) \int_{Q_\ell^k} f(x)\gamma_t(x) \mathrm{d}\rho_t(x) \mathrm{d}t \\
&= \int_0^{2r} \dot{\varphi}(t) \int_0^t \int_{\tilde{F}(\xi)} f(x)\gamma_\xi(x) \mathrm{d}\mathcal{H}^{N-2}(x) \mathrm{d}\xi \, \mathrm{d}t \text{ by (87)} \\
&= -\int_0^{2r} \varphi(t) \int_{\tilde{F}(t)} f(x)\gamma_t(x) \mathrm{d}\mathcal{H}^{N-2}(x) \mathrm{d}t.
\end{aligned}$$

So we see that in the cube Q_ℓ^k the pair $(\tilde{C}, \tilde{F})$ satisfies the front representation with front speed $v(x, t) = \gamma_t(x)$.

Part V Now, instead of taking single slices of the cube moving at speed of 1, we will take slices in a way that allows us to approximate the optimal front speed. By definition of $\mathscr{C}$, for any $\delta > 0$ we can choose $v^* \in (0, \infty)$ such that

$$\frac{\psi(v^*)}{v^*} \leqq \mathscr{C}(1+\delta).$$

Also, by the continuity of ψ, we can further take η small enough so that if $v^* < v_0 < v^* \frac{1}{1-\eta}$ we have

$$\frac{\psi(v_0)}{v_0} \leqq \mathscr{C}(1+\delta),$$

and hence

$$\psi(v_0) \leqq (1+\delta)\mathscr{C} v_0 \quad \text{when } v^* < v_0 < v^* \frac{1}{1-\eta}. \tag{89}$$

Set

$$l^{\min} := \frac{(2r)^{N-1}}{v^* \Delta t}$$

and

$$\tilde{l} := \frac{l^{\min}}{(2r)^{N-2}} = \frac{(2r)}{v^* \Delta t}.$$

We will employ the following notation:

- $\lfloor \tilde{l} \rfloor$ - the greatest integer less than or equal to $\tilde{l}$
- $\lceil \tilde{l} \rceil$ - the least integer that is greater than or equal to $\tilde{l}$
- $\{\tilde{l}\}$ - the fractional part of $\tilde{l}$
- $t^* := (1 - \{\tilde{l}\})\Delta t + t_1$.

First, in the interval $[t_1, t^*]$, set

$$\lambda_* := v^* \lfloor \tilde{l} \rfloor (t^* - t_1).$$

Then, for $m \in \mathbb{N}$, $m \leqq \lfloor \tilde{l} \rfloor$, define

$$S_m(t) := \frac{\lambda_*(m-1)}{\lfloor \tilde{l} \rfloor} + v^*(t - t_1).$$

We perform a similar construction in $(t^*, t_2]$, namely set

$$\lambda^* := v^* \lceil \tilde{l} \rceil (t_2 - t^*),$$

and for $m \in \mathbb{N}$, $m \leqq \lceil \tilde{l} \rceil$, define

$$S_m(t) := \frac{\lambda^*(m-1)}{\lceil \tilde{l} \rceil} + v^*(t - t_1).$$

Then, define

$$S(t) := \begin{cases} \bigcup_{\substack{m\in\mathbb{N} \\ m\leqq\lfloor\tilde{l}\rfloor}} \{S_m(t)\} & \text{if } t \in [t_1, t^*] \\ \bigcup_{\substack{m\in\mathbb{N} \\ m\leqq\lceil\tilde{l}\rceil}} \{S_m(t)\} & \text{if } t \in (t^*, t_2] \end{cases}$$

The function S then maps t to the set of points in $\mathbb{R}$ where we want to take slices of the cube at time t. Note that

$$\bigcup_{t\in[t_1,t_2]} S(t) = [0, 2r],$$

and further that every $\xi \in [0, 2r]$ belongs to $S(t)$ for only one t. Now define, for $t \in [t_1, t_2]$,

$$F_\eta^{k,\ell}(t) := \tilde{F}\left(S\left(t\right)\right)$$

$$C_\eta^{k,\ell}(t) := \begin{cases} \bigcup_{\substack{m\in\mathbb{N} \\ m\leqq\lfloor\tilde{l}\rfloor}} \{y \in G_\ell^k : S_m(t_1) \leqq y^N \leqq S_m(t)\} & \text{if } t \in [t_1, t^*] \\ \{y \in G_\ell^k : y^N \leqq \lambda^*\} \cup \bigcup_{\substack{m\in\mathbb{N} \\ m\leqq\lceil\tilde{l}\rceil}} \{y \in G_\ell^k : S_m(t^*) \leqq y^N \leqq S_m(t)\} & \text{if } t \in (t^*, t_2] \end{cases}$$

Note that by construction of the slices, $C_\eta^{k,\ell}(t) = G_\ell^k$. Then, in a manner similar to above, define the family of measures ρ_t^v, $t \in [t_1, t_2]$, by setting, for any Borel $A \subset Q_\ell^k$

$$\rho_t^v(A) := \int_{t_1}^{t_2} v^* \mathcal{H}^{N-2}\left(F_\eta^{k,\ell}(t) \cap A\right) \mathrm{d}t.$$

For reasons similar to those used for the measures ρ_t, these measures are all well defined Radon measures. Now, by applying the change of variables

$$\int_0^{\frac{(2r)}{\lambda}} \mathcal{H}^{N-2}(\tilde{F}(\lambda t))\mathrm{d}t = \lambda \int_0^{2r} \mathcal{H}^{N-2}(\tilde{F}(t))\mathrm{d}t,$$

to each of the slices individually, we find that

$$\rho_{t_2}^v(Q_\ell^k) = \rho_{2r}(Q_\ell^k), \tag{90}$$

however note that such an equality does not necessarily hold at any other time in $t \in [t_1, t_2]$. Also, with a similar restriction to one slice regions, the previous argument for the measures ρ_t can be modified to prove that $\forall t \in [t_1, t_2]$, we have that for any ball $B \subset Q_\ell^k$

$$(1-\eta)|D\chi_{E_x}|\left(C_\eta^{k,\ell}(t) \cap B\right) \leqq \rho_t^v(B) \leqq |D\chi_{E_x}|\left(C_\eta^{k,\ell}(t) \cap B\right). \tag{91}$$

This means that $\forall t \in [t_1, t_2]$,

$$|D\chi_{E_x}|\lfloor C_\eta^{k,\ell}(t) << \rho_t^v,$$

and that the densities

$$\gamma_t^v(x) := \frac{\mathrm{d}(|D\chi_{E_x}|\lfloor C_\eta^{k,\ell}(t))}{\mathrm{d}\rho_t^v}(x)$$

exist $\forall t \in [t_1, t_2]$, ρ_t^v-almost every $x \in Q_\ell^k$, and satisfy the uniform bounds

$$1 \leqq \gamma_t^v \leqq \frac{1}{1-\eta}.$$

Therefore, for $\varphi \in C_0^1([t_1, t_2])$ and $f \in C_0(\Omega')$,

$$\begin{aligned}\int_{t_1}^{t_2} \dot{\varphi}(t) \int_{C_\eta^{k,\ell}(t)\cap Q(x,r)} & f(x)\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}t \\ &= -\int_{t_1}^{t_2} \varphi(t) \int_{F_\eta^{k,\ell}(t)} f(x)v^*\gamma_t^v(x)\mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t.\end{aligned}$$

Since for any $f \in C_0(\Omega')$, the map

$$t \mapsto \int_{C_\eta^{k,\ell}(t)\setminus Q_\ell^k} f(x)\mathrm{d}\mathcal{H}^{N-1}(x)$$

is constant in $[t_1, t_2]$, we have

$$\int_{t_1}^{t_2} \dot{\varphi}(t) \int_{C_\eta^{k,\ell}(t)\setminus Q_\ell^k} f(x)\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}t = 0,$$

for any $\varphi \in C_0^1([t_1, t_2])$. Therefore we have that for $\varphi \in C_0^1([t_1, t_2])$ and $f \in C_0(\Omega')$

$$\int_{t_1}^{t_2} \dot{\varphi}(t) \int_{C_\eta^{k,\ell}(t)} f(x)\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}t = -\int_{t_1}^{t_2} \varphi(t) \int_{F_\eta^{k,\ell}(t)} f(x)v^*\gamma_t^v(x)\mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t.$$

Thus, in each time interval Y_ℓ^k the pair $(C_\eta^{k,\ell}, F_\eta^{k,\ell})$ satisfies the front representation, with front velocity $v_\eta^{k,\ell}(x,t) = \gamma_t^v(x)v^*$. Employing the uniform bounds on γ_t^v and (89), we have the following upper bound on the dissipation for the trajectory in the cube for η small enough:

$$\begin{aligned}\int_{t_1}^{t_2}\int_{F_\eta^{k,\ell}(t)} \psi(v_\eta^{k,\ell}(x,t))\mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t &= \int_{t_1}^{t_2}\int_{F_\eta^{k,\ell}(t)} \psi(v^*\gamma_t(x))d\mathcal{H}^{N-2}(x)dt \\ &\leqq (1+\delta)\mathscr{C}\int_{t_1}^{t_2}\int_{F_\eta^{k,\ell}(t)} v^*\gamma_t(x)\mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t \\ &= (1+\delta)\mathscr{C}\int_{t_1}^{t_2}\int_{F_\eta^{k,\ell}(t)} v_\eta^{k,\ell}(x,t)\mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t.\end{aligned}$$

Part VI Repeat this construction in each cube Q_l^k during the time interval Y_l^k, and in this way define the functions C_η^k and F_η^k for $\mathcal{L}^1$ almost every $t \in I_k$. From Part

V we have that the front representation formula holds in each time interval, which means that for $\varphi \in C^1_0(I_k)$ and $f \in C_0(\Omega')$

$$\int_{t_1}^{t_2} \dot{\varphi}(t) \int_{C_\eta^{k,\ell}(t)} f(x) \mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}t = -\int_{t_1}^{t_2} \varphi(t) \int_{F_\eta^{k,\ell}(t)} f(x) v^* \gamma_t^v(x) \mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t + \int_{F_\eta^{k,\ell}(t_2)} f \mathrm{d}\mathcal{H}^{N-2}(x) - \int_{F_\eta^{k,\ell}(t_1)} f \mathrm{d}\mathcal{H}^{N-2}(x)$$

where the boundary terms are the traces of the $L^1(I_k)$ function

$$t \mapsto \int_{F_\eta^{k,\ell}(t)} f \mathrm{d}\mathcal{H}^{N-2}(x). \tag{92}$$

Thus, by linearity of the integral we sum over all of the intervals Y_l^k, and use the fact that there are no jumps in the traces of the function (92) at the endpoints of each interval Y_l^k, to see that the front representation holds for C_η^k and F_η^k in I_k, that is, for $\varphi \in C^1_0(I_k)$ and $f \in C_0(\Omega')$

$$\int_{t_1}^{t_2} \dot{\varphi}(t) \int_{C_\eta^k(t)} f(x) \mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}t = -\int_{t_1}^{t_2} \varphi(t) \int_{F_\eta^k(t)} f(x) v_\eta^k \mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t.$$

Now, define A_{k+1} by setting

$$A_{k+1} := A_k \backslash C_\eta^k(I_k),$$

where by $C(I_k)$ we mean the C_η^k image of the set I_k. By the above we have

$$\begin{aligned} \mathcal{H}^{N-1}(A_{k+1}) &\leqq \sum_{l=1}^{\infty} \frac{2}{k} \eta (2 r_l^k)^{N-1} \\ &\leqq \frac{2}{k} \frac{1}{1-\frac{\eta}{k}} \mathcal{H}^{N-1}(A_k). \end{aligned} \tag{93}$$

Then, repeat the above construction for each A_k on the time interval I_k, $k > 1$, to define the functions C_η and F_η on all of $[a, b]$. We apply a similar argument to the above to show that the pair (C_η, F_η) satisfies the front representation formula in $[0, T]$. Now, since $\{A_k\}_{k=1}^{\infty}$ is a decreasing sequence of $\mathcal{H}^{N-1}$ measurable sets and $\mathcal{H}^{N-1}(A_k) < \infty$, by (93)

$$\mathcal{H}^{N-1}\left(C_\eta(b) \backslash \Gamma\right) = 0.$$

Since all of the time intervals are disjoint and cover almost all of $[a, b]$, we have that

$$\begin{aligned} \int_a^b \int_{F_\eta(t)} \psi(v(x,t)) \mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t &\leqq (1+\delta)\mathscr{C} \int_a^b \int_{F_\eta(t)} v_\eta(x,t) \mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t \\ &= (1+\delta)\mathscr{C}\mathcal{H}^{N-1}(\Gamma). \end{aligned}$$

$\square$

Now we prove the relaxation theorem.

Theorem 6.13. *Let* $\psi : [0, \infty) \to [0, \infty)$ *be continuous and*

$$I_\epsilon[q] := \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \int_{F(t)} \psi(v(x,t))\mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t \tag{94}$$

for $q = (u, C) \in \mathcal{T}$. *Then* I_ϵ^*, *the lower semicontinuous envelope in* $\mathcal{T}^*$ *of the functional* I_ϵ, *with respect to the convergence defined by* (62), *is given by*

$$I_\epsilon^*[q] = \mathscr{C} \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \mathrm{d}\mu(t), \tag{95}$$

where $q = (u, C) \in \mathcal{T}^*$, C^* *is the minimal crack set trajectory associated to* q, μ *is the weak derivative of* $t \mapsto \mathcal{H}^{N-1}(C^*(t))$, *and*

$$\mathscr{C} := \inf_{s\in(0,\infty)} \frac{\psi(s)}{s},$$

Proof. The proof proceeds as follows. First, we use the results of Theorem 6.12 to construct a sequence $\{q_i\}_{i=1}^\infty \subset \mathcal{T}$ such that $q_i \to q$ and whose energies converge to the right-hand side of (95). Then we will combine this construction and the lower semicontinuity of the right-hand side of (95) to complete the proof.

Let $q = (u, C) \in \mathcal{T}^*$ with associated C^*. We construct a sequence $\{q_j\}_{j=1}^\infty \subset \mathcal{T}$ that converges to q and achieves the lower bound in the limit through the following. Let $\mathscr{D}$ be a countable dense subset of $[0, T]$ that generates q and contains the times 0 and T. For each $j \in \mathbb{N}$, choose

$$D_j := \left\{0 = t_0^j < t_1^j < \cdots < t_j^j = T\right\} \subset \mathscr{D}$$

such that $\{D_j\}$ is an increasing sequence of nested sets and

$$\mathscr{D} = \bigcup_{j=1}^\infty D_j.$$

Now, fix $j \in \mathbb{N}$. By definition of $\mathcal{T}^*$, for each $t \in [0, T]$ $u(\cdot, t) \in SBV_p(\Omega')$ where $p > 1$. Also, since $\mathscr{D}$ generates q then for every $t \in [0, T]$

$$C^*(t) \tilde{=} C_0 \cup \bigcup_{\substack{\tau \leqq t \\ \tau\in\mathscr{D}}} S(u(\tau)).$$

Since $\mathcal{H}^{N-1}(C^*(t)) \leqq \mathcal{H}^{N-1}(C^*(T)) < \infty$ for all $t \in [0, T]$, we can apply Lemma 6.10 and Theorem 6.12, so that for each interval $[t_k^j, t_{k+1}^j]$, $k = 0, \ldots, j-1$, we can choose a pair (C_k^j, F_k^j) satisfying the front representation and so that

$$C_k^j(t_{k+1}^j)\backslash C_k^j(t_k^j) \tilde{=} C^*(t_{k+1}^j)\backslash C^*(t_k^j) \tag{96}$$

and

$$\int_{t_k^j}^{t_{k+1}^j} \int_{F_k^j(t)} \psi\left(v_j(x,t)\right) \mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t \leqq \left(1+\frac{1}{j}\right) \mathcal{H}^{N-1}\left(C^*(t_{k+1}^j) \backslash C^*(t_k^j)\right)$$
$$= \left(1+\frac{1}{j}\right) \int_{t_k^j}^{t_{k+1}^j} \mathrm{d}|D\mathcal{H}^{N-1}(C^*(t))|. \tag{97}$$

Repeat this process for each $k = 0, \ldots, j-1$, and then define $\{q_j = (u_j, C_j)\}_{j=1}^{\infty}$ by setting

$$u_j(t) := \begin{cases} u(t_k^j) & \text{for } t \in [t_k^j, t_{k+1}^j) \\ u(T) & \text{for } t = T \end{cases}$$

and

$$C_j(t) := \begin{cases} C_k^j(t_k^j) & \text{for } t \in [t_k^j, t_{k+1}^j) \\ C^*(T) & \text{for } t = T. \end{cases}$$

Clearly we have for each $t \in \mathscr{D}$

$$u_j(\cdot, t) \to u(\cdot, t),$$

in fact for any such t there is an $M \in \mathbb{N}$ such that for all $j > M$, $u_j(\cdot, t) \equiv u(\cdot, t)$. Then, we have the upper bound

$$\begin{aligned} I_\epsilon[q_j] &= \sum_{k=0}^{j-1} \int_{t_k^j}^{t_{k+1}^j} \mathrm{e}^{-\frac{t}{\epsilon}} \int_{F_j(t)} \psi\left(v_j(x,t)\right) \mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t \\ &\leqq \left(1+\frac{1}{j}\right) \mathscr{C} \sum_{k=0}^{j-1} \int_{t_k^j}^{t_{k+1}^j} \mathrm{e}^{-\frac{t_k^j}{\epsilon}} \mathrm{d}|D\mathcal{H}^{N-1}(C^*(t))|, \end{aligned}$$

and the lower bound

$$\begin{aligned} I_\epsilon[q_j] &= \sum_{k=0}^{j-1} \int_{t_k^j}^{t_{k+1}^j} \mathrm{e}^{-\frac{t}{\epsilon}} \int_{F_j(t)} \psi\left(v_j(x,t)\right) \mathrm{d}\mathcal{H}^{N-2}(x)\mathrm{d}t \\ &\geqq \mathscr{C} \sum_{k=0}^{j-1} \int_{t_k^j}^{t_{k+1}^j} \mathrm{e}^{-\frac{t_k^{j+1}}{\epsilon}} \mathrm{d}|D\mathcal{H}^{N-1}(C^*(t))|. \end{aligned}$$

Thus, we have

$$I_\epsilon[q_j] \to \mathscr{C} \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \mathrm{d}|D\mathcal{H}^{N-1}(C^*(t))| \quad \text{as } j \to \infty. \tag{98}$$

We now combine the results above to complete the proof. For any $q = (u, C) \in \mathcal{T}^*$ with associated C^*, from Remark 6.11 and the sequence constructed in Part I, we have that

$$\mathscr{C} \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \mathrm{d}\mu \geqq I_\epsilon^*[q]. \tag{99}$$

The other inequality

$$I_\epsilon^*[q] \leqq \mathscr{C} \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \mathrm{d}\mu \leqq I_\epsilon^*[q] \tag{100}$$

follows from the following. For any $\{q_i\}_{i=1}^\infty \subset \mathcal{T}$ such that $q_i \to q$. we again combine Remark 6.11 and Part I to construct a sequence $\tilde{q}_i$ so that $\tilde{q}_i \to q$ with

$$\liminf_{i\to\infty} \mathscr{C} \int_0^T \mathrm{e}^{-\frac{t}{\epsilon}} \mathrm{d}\mu_i = \lim_{j\to\infty} I_\epsilon[\tilde{q}_j],$$

which combined with Lemma 6.9 gives (100). $\square$

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