Chapter 1

Numerical Methods for Structural Dynamics:

1.1 Introduction:

One looking at previous works done on the subject treated here may be surprised to see that models used for the structure part vary enourmously in their complexity and range of applicability. In fact, the assumptions made depends on many criteria, namely the shape, the dimension and the material. When the subject is rather oriented towards fluid dynamics than structural dynamics, many prefers, as a first step, to consider simple structural models. By simple we mean unidimensional linear elastic models.

1.2 Linear Elastic Model:

1.2.1 Introduction:

The use of linear elastic models depend on the following basic assumptions regarding the structure :

- the structure is thin : $h << R$,
- the strains are small everywhere, although large deflexions are admitted,
- the state of stress can be considered plane.

1.2.2 Thin plate model:

We consider a thin elastic flat plate of lenght L with infinite aspect ratio and having its undeflected surface in the x,y-plane. The governing equation for the motion of a thin plate is expressed as following:

$$m\frac{\partial^2 w(x,y,t)}{\partial t^2} + D\nabla^4 w(x,y,t) = f(x,y,t) \tag{1.1}$$

with

$$m = \rho h$$

and

$$D = \frac{Eh^3}{12(1-\nu^2}$$

If we consider that the longitudinal displacements can be neglected due to the plate's thinness, the equation (1.1) take the form:

$$m\frac{\partial^2 w(x,t)}{\partial t^2} + D\frac{\partial^4 w(x,t)}{\partial x^4} = f(x,t) \tag{1.2}$$

The boundary conditions expressing the fact that the plate is simply supported at the edges, see figure **??**, are given by:

$$y(x,t)|_{x=0,L} = 0 \quad , \quad \frac{\partial^2 y(x,t)}{\partial x^2}|_{x=0,L} = 0 \tag{1.3}$$

1.2.3 Spatial Discretisation:

Finite Differences Approach:

In this section we describe the use of the Finite Differences approach to discretise the mathematical model described in the previous section. We consider that the length of the plate L is represented by n equidistant nodes denoted by L_i and the space between the nodes is denoted by Δx. The 4th order spatial derivative is then discretised at y_i as follows:

$$\frac{\partial^4 y(x,t)}{\partial x^4}|_i = \frac{y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}}{\Delta x^4} + \mathcal{O}(\Delta x^2) \tag{1.4}$$

Replacing the spatial derivative by its discretisation (1.4) in the equation (1.2) we obtain the following ordinary differential equation:

$$m\frac{\partial^2 y_i}{\partial t^2} + D\frac{y_{i-2} - 4y_{i-1} + 6y_i - 4y_{i+1} + y_{i+2}}{\Delta x^4} = f_i \tag{1.5}$$

Using the central difference for the 2nd order boundary condition, the discretisation of the boundary conditions (1.3) is given by:

$$y_0 = y_n = 0$$

$$y_{-1} = -y_1$$

$$y_{n+1} = -y_{n-1}$$

With these boundary conditions we can express the equation (1.5) in the matrix form as:

$$[M]\ddot{y}(t) + [K]y(t) = f \tag{1.6}$$

$[M]$ denotes the mass matrix given by :

$$[M] = m[I]$$

$[K]$ denotes the stiffness matrix given by :

$$[K] = \frac{D}{\Delta x^4}\begin{pmatrix} 5 & -4 & 1 & & & \\ -4 & 6 & . & . & . & \\ 1 & . & . & . & . & \\ & . & . & . & . & 1 \\ & & . & . & 6 & -4 \\ & & & 1 & -4 & 5 \end{pmatrix}$$

y denotes vector of vertical displacement at the nodes and the double dot denotes the double derivative or acceleration. f is the force vector composed by local forces at the nodes. The size of vectors and matrices is $(n-2)$ and $(n-2)^2$ respectively.

Finite Elements Approach:

The weak form is obtained by multipying the equation by a weight function $\omega(x)$ and integrate the resulting equation over the domain, which leads to :

$$\int_x m\omega(x)\frac{\partial^2 y(x,t)}{\partial t^2}\,dx + \int_x D\omega(x)\frac{\partial^4 y(x,t)}{\partial x^4}\,dx = \int_x \omega(x) f_x\,dx \tag{1.7}$$

We use the integration by parts in order to simplify the 4th order partial derivative appearing in the second integral of the previous equation (1.7):

$$\int_x D\omega(x)\frac{\partial^4 y(x,t)}{\partial x^4}\,dx = [D\omega(x)\frac{\partial^3 y(x,t)}{\partial x^3}]_0^L - \int_x D\frac{\partial \omega(x)}{\partial x}\frac{\partial^3 y(x,t)}{\partial x^3}\,dx$$

and

$$\int_x D\frac{\partial \omega(x)}{\partial x}\frac{\partial^3 y(x,t)}{\partial x^3}\,dx = [D\frac{\partial \omega(x)}{\partial x})\frac{\partial^2 y(x,t)}{\partial x^2}]_0^L - \int_x D\frac{\partial^2 \omega(x)}{\partial x^2}\frac{\partial^2 y(x,t)}{\partial x^2}\,dx$$

We impose the following condition on $w(x)$:

$$y(x,t)|_{x=0,L} = 0 \Rightarrow w(x)|_{x=0,L} = 0$$

which leads to the simplified form of equation (1.7):

$$\int_x m\omega(x)\frac{\partial^2 y(x,t)}{\partial t^2}\,dx + \int_x D\frac{\partial^2 \omega(x)}{\partial x^2}\frac{\partial^2 y(x,t)}{\partial x^2}\,dx = \int_x \omega(x) f_x\,dx \tag{1.8}$$

The domain is considered one dimension, according to this the plate is subdivised into $n-1$ itervals or elements, i.e n nodes. In order to describe the diplacement by a 3rd order function we need to define two degree of freedom per nodes, namely the displacment and the rotation angle, which leeds to 4 degree of freedom per element. If we consider, as shown in figure 2, one element defined by two nodes, x_i and x_{i+1}. The displacments at these nodes are denoted by y_i and y_{i+1} and the rotation angles by ω_i and ω_{i+1}.

The coordinates are transformed into natural ones which are bounded by $-1 < \eta < 1$, we may then express the displacment $Y(\eta)$ as a function of the four degrees of freedom:

$$Y(\eta) = y^T N(\eta)$$

Where

$$y = \begin{pmatrix} y_i \\ \omega_i \\ y_{i+1} \\ \omega_{i+1} \end{pmatrix}, N(\eta) =$$

With the vector $N(\eta)$ containing the third order interpolation functions of the element. We use the Galerkin approach, we may then write:

$$\omega(\eta) = N(\eta)$$

We need also to transform the partial derivatives with respect to x into partial derivatives with respect to η as the following:

$$\frac{\partial}{\partial x} = \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{2}{\Delta x}\frac{\partial}{\partial \eta}$$

1.2.4 Eigenvalue Analysis:

The aim of the eigenvalues analysis is to validate the model proposed by comparing the analytical eigenfrequencies to the numerical ones. During the eigenvalues analysis we don't consider the force vector; we may then consider the following equation:

$$[M]\ddot{y}(t) + [K]y(t) = 0$$

In order to obtain a generalized eigenvalue problem we substitute the harmonic time dependence [I.3.b] in the equation [I.3.a]:

$$y(t) = ye^{i\omega t}$$

$$([K] - \lambda[M])y = 0$$

1.2.5 Time Discretisation:

The Newmark methods family is one of several direct integration methods available for second order hyperbolic equations and is widely used for this purpose. In this scheme, the following assumptions are considered for both the function and its first derivative:

$$y^{n+1} = y^t + \Delta t\dot{y}^n + \Delta t^2[(\frac{1}{2} - \alpha)\ddot{y}^n + \alpha\ddot{y}^{n+1}] \tag{1.9}$$

and

$$\dot{y}^{n+1} = \dot{y}^n + \Delta t[(\frac{1}{2} - \beta)\ddot{y}^n + \beta\ddot{y}^{n+1}] \tag{1.10}$$

Where α and β denotes parameters controling the stability and accuracy of the method.

The constant-average acceleration method had been proposed originally by Newmark who states that this scheme in unconditionnally stable for $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{2}$. The acceleration vector $\ddot{y}(t)$ is defined as a constant average of the vectors at t^n and t^{n+1} :

$$\ddot{y}(t) = \frac{1}{2}(\ddot{y}^n + \ddot{y}^{n+1}) \tag{1.11}$$

In this case we may rewrite the equations 1.9 and 1.10 as the following:

$$y^{n+1} = y^n + \Delta t\dot{y}^n + \frac{\Delta t^2}{4}[(\ddot{y}^n + \ddot{y}^{n+1}] \tag{1.12}$$

and

$$\dot{y}^{n+1} = \dot{y}^n + \frac{\Delta t}{2}[(\ddot{y}^n + \ddot{y}^{n+1}] \tag{1.13}$$

The equation 1.12 is rearranged to express the acceleration vector at $n + 1$:

$$\ddot{y}^{n+1} = \frac{4}{\Delta t^2}(y^{n+1} - y^n) - \frac{4}{\Delta t}\dot{y}^n - \ddot{y}^n \tag{1.14}$$

And using the equations 1.13 and 1.14 the velocity vector at $n+1$ is expressed by :

$$\dot{y}^{n+1} = \frac{2}{\Delta t}(y^{n+1} - y^n) - \dot{y}^n \tag{1.15}$$

Substitution of equation 1.14 in the equation of motion **??** at $n+1$ gives the equation for the displacment vector:

$$y^{n+1} = [\frac{4}{\Delta t^2}M + K] + (F^{n+1} + M(\frac{4}{\Delta t^2}y^n + \frac{4}{\Delta t}\dot{y}^n + \ddot{y}^n)) \tag{1.16}$$

The solution process begin at time $t = 0$, i.e. $n = 0$ by a given inital vector y^0 and inital vector $\dot{y}^0$. The initial acceleration vector $\ddot{y}^0$ is then determined via the equation of motion **??** at $n = 0$:

$$\ddot{y}^0 = M^{-1}(F^0 - Ky^0) \tag{1.17}$$

The displacment vector at $n+1$ is obtained using the equation 1.16, then the velocity and acceleration vectors are calculated by the equations 1.15 and 1.14 respectively.

Algorithm:
Initialisation:

1. Form stiffness matrix K and mass matrix M
2. Calculate effective loads F at $t = 0$
3. Initialise y^0, $\dot{y}^0$, $\ddot{y}^0$
4. Select time step Δt
5. Form effective matrix $A = \frac{4}{\Delta t^2}M + K$
6. Inverse matrix A

Iteration:

1. Calculate effective loads F at $t + \Delta t$
2. Calculate displacements at $t + \Delta t$
3. Calculate velocities and accelerations at $t + \Delta t$
4. Check convergence