Statement:

$$P(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}$$

Proof (by induction):

Initial steps:

$$P(0) = (-1)^0 \binom{0-0}{0} x^{0-2\cdot 0} = 1 \tag{1}$$

$$P(1) = (-1)^0 \binom{1-0}{0} x^{1-2\cdot 0} = x \tag{2}$$

Induction step:

$$P(n+1) = xP(n) - P(n-1) \tag{3}$$

$$= x \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k} \quad - \quad \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-k-1}{k} x^{n-2k-1} \tag{4}$$

$$= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k+1} \quad - \quad \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n-k-1}{k} x^{n-2k-1} \tag{5}$$

Now we proceed on two distinct cases:

- When n is even: $\quad (\lfloor \frac{n}{2} \rfloor = \frac{n}{2},\ \lfloor \frac{n-1}{2} \rfloor = \frac{n}{2} - 1,\ \lfloor \frac{n+1}{2} \rfloor = \frac{n}{2})$

  Taking out the first term ($k = 0$) of the first sum, in order to eventually group terms of the same degree together:

$$P(n+1) \ = \ x^{n+1} \ + \ \sum_{k=1}^{\frac{n}{2}} (-1)^k \binom{n-k}{k} x^{n-2k+1} \ - \ \sum_{k=0}^{\frac{n}{2}-1} (-1)^k \binom{n-k-1}{k} x^{n-2k-1} \tag{6}$$

$$= \ x^{n+1} \ + \ \sum_{k=1}^{\frac{n}{2}} (-1)^k \left(\binom{n-k}{k} + \binom{n-k}{k-1} \right) x^{n-2k+1} \tag{7}$$

$$= \ x^{n+1} \ + \ \sum_{k=1}^{\frac{n}{2}} (-1)^k \binom{n-k+1}{k} x^{n-2k+1} \tag{8}$$

$$= \ \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \binom{n-k+1}{k} x^{n-2k+1} \tag{9}$$

  as expected.

- When n is odd: $\quad (\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2},\ \lfloor \frac{n-1}{2} \rfloor = \frac{n-1}{2},\ \lfloor \frac{n+1}{2} \rfloor = \frac{n-1}{2} + 1)$

  Here we will take out the first term ($k = 0$) of the first sum, and the last term ($k = \frac{n-1}{2}$) of the second sum:

$$\begin{aligned}
P(n+1) &= x^{n+1} - (-1)^{\frac{n-1}{2}} \\
&\quad + \sum_{k=1}^{\frac{n-1}{2}} (-1)^k \binom{n-k}{k} x^{n-2k+1} - \sum_{k=0}^{\frac{n-1}{2}-1} (-1)^k \binom{n-k-1}{k} x^{n-2k-1} \quad (10) \\
&= x^{n+1} - (-1)^{\frac{n-1}{2}} + \sum_{k=1}^{\frac{n-1}{2}} (-1)^k \left(\binom{n-k}{k} + \binom{n-k}{k-1} \right) x^{n-2k+1} \quad (11) \\
&= x^{n+1} + (-1)^{\frac{n-1}{2}+1} + \sum_{k=1}^{\frac{n-1}{2}} (-1)^k \binom{n-k+1}{k} x^{n-2k+1} \quad (12) \\
&= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} (-1)^k \binom{n-k+1}{k} x^{n-2k+1} \quad (13)
\end{aligned}$$

as before.