THOMAS ALGORITHM

We explain how systems resulting from implicit schemes with three space points $(i-1), i, (i+1)$ can be solved. We can write them in the form :

$$a_i u_{i-1} + b_i u_i + c_i u_{i+1} = f_i \quad \forall \quad i = 1, \cdots, N_x - 1$$

With boundary conditions: $u_{i=0} = B_0$ and $u_{i=N_x} = B_{N_x}$ What results is the following matrix system :

$$\begin{pmatrix} 1 & 0 & 0 & . & . & . & . \\ a_1 & b_1 & c_1 & 0 & . & . & . \\ 0 & a_2 & b_2 & c_2 & 0 & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0 \\ . & . & . & 0 & a_{N_x-1} & b_{N_x-1} & c_{N_x-1} \\ . & . & . & . & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ . \\ . \\ . \\ u_{N_x-1} \\ u_{N_x} \end{pmatrix} = \begin{pmatrix} B_0 \\ f_1 \\ . \\ . \\ . \\ f_{N_x-1} \\ B_{N_x} \end{pmatrix}$$

The boundary conditions can be gathered in the second member as:

$$\begin{pmatrix} b_1 & c_1 & 0 & . & . & . \\ a_2 & b_2 & c_2 & 0 & . & . \\ 0 & . & . & . & . & . \\ . & . & . & . & . & 0 \\ . & . & . & . & . & c_{N_x-2} \\ . & . & . & 0 & a_{N_x-1} & b_{N_x-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ u_{N_x-2} \\ u_{N_x-1} \end{pmatrix} = \begin{pmatrix} f_1 - a_1 B_0 \\ f_2 \\ . \\ . \\ f_{N_x-2} \\ f_{N_x-1} - c_{N_x-1} B_{N_x} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ . \\ . \\ F_{N_x-2} \\ F_{N_x-1} \end{pmatrix}$$

The matrix obtained possesses three diagonals, the system is then solved by the Thomas Algorithm. It is done in two distinct steps. The first consists of "removing" the coefficients a_is through a forward sweep. The new system with the coefficient matrix having two diagonals now looks like:

$$\begin{pmatrix} b'_1 & c'_1 & 0 & . & . & . \\ 0 & b'_2 & c'_2 & 0 & . & . \\ 0 & . & . & . & . & . \\ . & . & . & . & . & 0 \\ . & . & . & 0 & b'_{N_x-2} & c'_{N_x-2} \\ . & . & . & 0 & 0 & b'_{N_x-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ . \\ . \\ u_{N_x-2} \\ u_{N_x-1} \end{pmatrix} = \begin{pmatrix} F'_1 \\ F'_2 \\ . \\ . \\ F'_{N_x-2} \\ F'_{N_x-1} \end{pmatrix}$$

And we have a new algebraic system defined by :
$b'_i u_i + c'_i u_{i+1} = F'_i \qquad \forall\ i = 1, \cdots, N_x - 2$
and
$b'_{N_x-1} u_{N_x-1} = {F'}_{N_x-1} \qquad \text{for} \quad i = N_x - 1$
with the new coefficients :

$$b'_1 = b_1 \quad , \quad F'_1 = F_1$$

$$c'_i = c_i \quad \forall\ i = 1, \cdots, N_x - 1$$

$$b'_i = b_i - c'_{i-1} \frac{a_i}{b'_{i-1}} \quad \forall i = 2 \cdots N_x - 1$$

$$F'_i = F_i - F'_{i-1} \frac{a_i}{b'_{i-1}} \quad \forall i = 2 \cdots N_x - 1$$

The second step consists of backward sweep of the new matrix in order to calculate the solution : We obtain initially

$$u_{N_x-1} = \frac{{F'}_{N_x-1}}{b'_{N_x-1}}$$

then, we calculate the solution as

$$u_i = \frac{F'_i - c'_i u_{i+1}}{b'_i} \quad \forall \quad i = N_x - 2, \cdots, 1$$