Extra axioms for Euclid.
We assume Euclid's postulates and common notions, as well as the following extra assumptions: separation properties, congruence axioms, existence of rigid motions, existence of intersections, and "equality of content" for polygons. In more detail, we assume:

"Obvious stuff":

I. Separation properties:
We assume each point on a line separates the line into two sides, and given two points A,B, the segment AB consists of all points which are on the same side of A as B, and also on the same side of B as A. A point C of this segment not equal to A or B, is said to be between A and B, and A and B are on opposite sides of C. For any three points on a line, exactly one of them is between the other two.

Given any line L, we assume it separates the plane into two non empty "convex" sides, i.e. that for any two points A,B on the same side of L, the segment AB lies entirely on that same side of L. Two points A,B lie on opposite sides of L if and only if the segment AB meets L.

Definition of Rays and angles, the inside of an angle:

A "ray" is a point p of a line L, plus the set of all points of L lying on one side of p. We say p is the "vertex".

An "angle" is two distinct rays with the same vertex plus a choice of a "side" of the angle as the "inside. If the rays form a line, both angles they form are "straight angles".

If they do not form a line, they determine one "proper angle" (less than a straight angle), and one "improper angle" (greater than a straight angle), as follows.

For each ray consider the halfplane it determines which contains the other ray. The common part of those two halfplanes is the inside of the proper angle. The union of the other two halfplanes is the inside of the improper angle.

When we speak of "the" angle formed by two rays, we usually mean the proper angle.

II.A Congruence of segments:
We assume a notion of congruence of segments, which is an equivalence relation, and that when we have congruent pairs of segments A1,B1; A2,B2;,...,An,Bn, with each Aj congruent to Bj, then the segment obtained by combining all the Aj together into one segment is congruent to the segment obtained by combining all the Bj together into one segment. Moreover, if C is between A and B on a line, then segment AC is not congruent to segment AB (the whole is not greater than the part).

IIB. Congruence of angles:

We have a notion of congruence of angles, which is an equivalence relation.

If two angles A,B have the same vertex, and the inside of the angle B is part of the inside of A, and if the angle B has at least one of its rays lying inside the angle A, then the angle B is smaller than the angle A, (the whole is greater than the part).

Also we can "add" angles, in the sense that if two angles have one common side, but disjoint insides, then we can combine them into one angle. If we have congruent pairs of angles A1,B1;A2,B2;....;An,Bn; and we combine the Aj into one angle, and the Bj into one angle, we obtain congruent angles A,B, as we did for segments.

Not quite so obvious stuff:

Definition: A "rigid motion" is a one to one correspondence of the plane with itself, taking lines to lines, segments to congruent segments, and angles to congruent angles.

III. Existence axioms for rigid motions (ERM):
1) For any two points, there is a rigid motion taking one point to the other. (think of translations.)
2) For any three points O,A,B with A,B different from O, there is a rigid motion fixing O and taking the ray OA to the ray OB. (think rotation.)
3) For any line L, there is a rigid motion fixing every point of L, and interchanging the two sides of L. (think reflection.)

We also add to axiom 1 of Euclid, that there is <u>only one</u> straight line segment joining any two distinct points.

Then we can prove SAS as follows:
If AB and DE are congruent segments, there is a rigid motion taking A to D, and B to E, and that if ABC and DEF are triangles with side |AB| = |DE|, and |BC| = |EF|, and angle ABC = angle DEF, then there is a rigid motion taking A to D, B to E, and C to F. Then |AC| = |DF| by the stronger axiom 1. Also angles BAC and BCA are equal respectively to angles EDF and EFD, since our motion preserved congruence of angles, and consequently the triangles ABC and DEF are congruent.

Then we can define congruence of any two figures to mean there is a rigid motion taking one to the other.

For Prop I.1, to construct an equilateral triangle with given side,
we need a **circle-circle intersection axiom:**

Define the "inside" of a circle with center P and point R on the circumference, as all points Q such that the segment PQ is shorter than the segment PR, and the outside of the circle as all points Q such that PQ is longer than PR.

IVA. We assume: (LC): If A is a point inside a circle and B is a point outside that circle, then the segment AB contains exactly one point of the circle.

IVB. We also assume: (CC): If one circle C1 contains a point inside another circle C2 and also a point outside circle C2, then the two circles meet on their circumferences.

For Pythagoras, we need a concept of "equal content" of plane figures.
V. There is an equivalence relation on polygons called "equal content", such that congruent polygons have equal content, and polygons that can be decomposed into congruent pieces (equidecomposable polygons) have equal content. Also if A and B are equidecomposable and A+S and B+T are equidecomposable, then S and T have equal content, i.e. figures obtained by subtracting figures of equal content also have equal content.
A figure properly contained inside another figure (i.e. such that some point interior to the larger figure is not in the smaller one), has smaller content (the whole is greater than the part).

This is used to prove Prop. I.39 (triangles on same base but of different "height" do not have equal content), and hence also Prop. I.48 (converse of Pythagoras: if a triangle has an angle which is not a right angle, then the square on that side does not have content equal that of the squares on the other two sides together).

Notice we are making sense of comparing sizes of segments and polygons without assigning any numbers to their "lengths" or "areas" so far. Later we will want to do this, but in fact the theory of number arose partly in response to the need to do this, so we want to see what properties those numbers should have. It will turn out that many more number systems will satisfy our needs than just the "real numbers". Thus we will encounter many more geometries and many more number systems by waiting to see what we need than by imposing the use of real numbers at the beginning. As remarked today, there will be number systems and hence also lines, with fewer points than there are real numbers, but also ones where lines have more points than the real numbers! Thus our theorems will apply to geometries where there are both "infinitely small" and "infinitely large" triangles and segments.