Formulation and Solution of a Relativistic Harmonic Oscillator

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There appear to be several forms of the relativistic harmonic oscillator published in the literature that may not agree. Curiously, the harmonic oscillator problem is not extensively treated in textbooks as might be expected. My interest in this subject arose from reading a Google file on the subject full of errors and subjective opinions. It certainly does reinforce idea of 'buyer beware' when taking net assertions as gospel.

My interest here is to properly define the relativistic harmonic oscillator problem and then solve it by some means.

The problem context is this: an oscillator consisting of a proper mass m oscillates along the x-axis in periodic or quasi-periodic motion. The two inertial reference frames are the laboratory frame in which the classical restoring force is kx, where x is the mass displacement from the origin, and the particle inertial frame in which the particle is at rest.

The classical limit is the well-known equation

$$m\ddot{x} + kx = 0 \tag{1.1}$$

In addition to the particle mass, the Hook's Law spring constant k appears as the only other parameter defining the problem.

As is well-known, the solutions to this second-order linear equations are sines and cosines oscillating at a frequency

$$\omega = \sqrt{\frac{k}{m}} \tag{1.2}$$

The selection of sines and cosines will depend on the initial conditions chosen.

Let us now generalize (1.1) to describe relativistic motion of the mass.

Textbooks[1] show that the covariant form of the equations of motion transform as 4-coordinates in special relativity. In any inertial system;

$$\begin{aligned} \frac{d\vec{p}}{dt} &= \vec{F} \\ \frac{dE}{dt} &= \vec{F}\cdot\vec{u} \end{aligned} \tag{1.3}$$

[1] C. Moller, The Theory of Relativity, Oxford at the Clarendon Press, 1952. p.73, et seq.

The first of these reveals the time derivative of the momentum equals the force and the second shows conservation of energy where u is the vector direction of the particle. In our case u has only an x component and describes the relative speed of the two inertial frames. (Special relativity holds because at any proper time t, one can always find a Lorentz transformation from the laboratory to the particle reference frame and vice versa since the meaning of the coordinates does not change. This situation of course is not true in general relativity where space-time depends on gravity.)

As emphasized above, (1.1) represents a description of the harmonic oscillator in the laboratory frame in which the origin of the restoring force will be placed at the origin. One could alternately describe the oscillator from the particle inertial frame by introducing the proper time τ as measured by a clock moving with the particle. Then the force must also be modified to account for the change in measuring rods and clocks when comparing the two inertial frames. The result is the equation of motion as described in the particle frame as the origin swings back and forth.

$$\frac{d\vec{p}}{d\tau} = \frac{\vec{F}}{\sqrt{1-\frac{u^2}{c^2}}} \qquad d\tau = dt\sqrt{1-\frac{u^2}{c^2}} \tag{1.4}$$

The second expression shows that a clock in the inertial frame appears to run slower than the proper clock.

Now we have a Lorentz covariant form of the equation of motion which can now be solved.

When written out in one dimension, the result in the laboratory frame is:

$$\frac{d}{dt}\left(\frac{mu}{\sqrt{1-\frac{u^2}{c^2}}}\right) + kx = 0; \quad u = \frac{dx}{dt} \tag{1.5}$$

Remember that m is the proper mass.

Let w be a non-dimensional length: x ω/c and $\tau = \omega\tau$ a non-dimensional time. After carrying out the indicated derivatives in (1.5) and noting that the acceleration is in the same direction as the force, we find the second order dimensionless equation:

$$\ddot{w} + \left(1-\dot{w}^2\right)^{\frac{3}{2}} w = 0\,. \tag{1.6}$$

It is clear from this form that the classical limit is easily recovered when dw/dτ is small everywhere compared to 1.

One constant of the motion is obtained by multiplying (1.8) through by dw/dτ to obtain a Hamilton-Jacoby form of the motion equation:

$$\dot{w}^2 + \left(1-\dot{w}^2\right)^{\frac{3}{2}} w^2 = w_0^2 \tag{1.7}$$

By examining the dimensionless variables, one can also see that the constant of the motion w is related to the maximum extension x_0 of the harmonic oscillator.

$$\begin{aligned} \left(\frac{\omega x_0}{c}\right)^2 &= w_0^2 \\ w_0 &= \sqrt{2}\left(\frac{k x_0^2}{2mc^2}\right)^{\frac{1}{2}} \end{aligned} \tag{1.8}$$

This expression reveals the relativistic regime: When the maximum potential energy becomes comparable to the rest energy of the particle, we find strong departures from the classical limit. Since the rest energies of most particles are in the MeV region, one appreciates that the nature of the relativistic departure will only happen to very strongly bound particles—such as harmonic motion of nuclear material. It is probable that very strongly excited 'liquid drop' nuclei might oscillate in the relativistic regime before breaking up. (This statement is true for MeV neutron-induced fission in the actinides. However for the odd nuclei, the fission threshold drops into the hundred volt region, hence the need to moderate fissile neutron energies in order to sustain a critical reaction in ^{235}U as in a nuclear reactor, for example).

It turns out that while (1.7) is helpful in understanding the problem, it is not particularly useful to set up a numerical solution. It is better to work with (1.6) by regarding it as a pair of first order differential equations. Then there are a number of ODE (ordinary differential equation) packages available (in MATLAB, say) to render a numeric solution.

The pair of equations useful in numerical ODE solvers is:

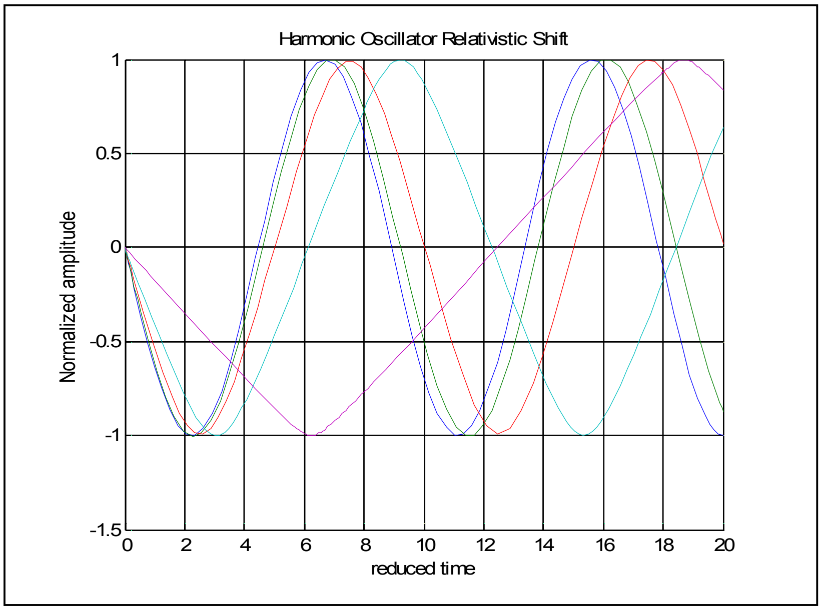
$$\begin{aligned} \frac{dw}{d\tau} &= y \\ \frac{dy}{d\tau} &= -\left(1-y^2\right)^{\frac{3}{2}} w \end{aligned} , \tag{1.9}$$

subject to the initial conditions, y(0) = 0, w(0) = w_0.

This pair of equations has been integrated using a standard MATLAB routine (ODE45) for τ extending through several periods.

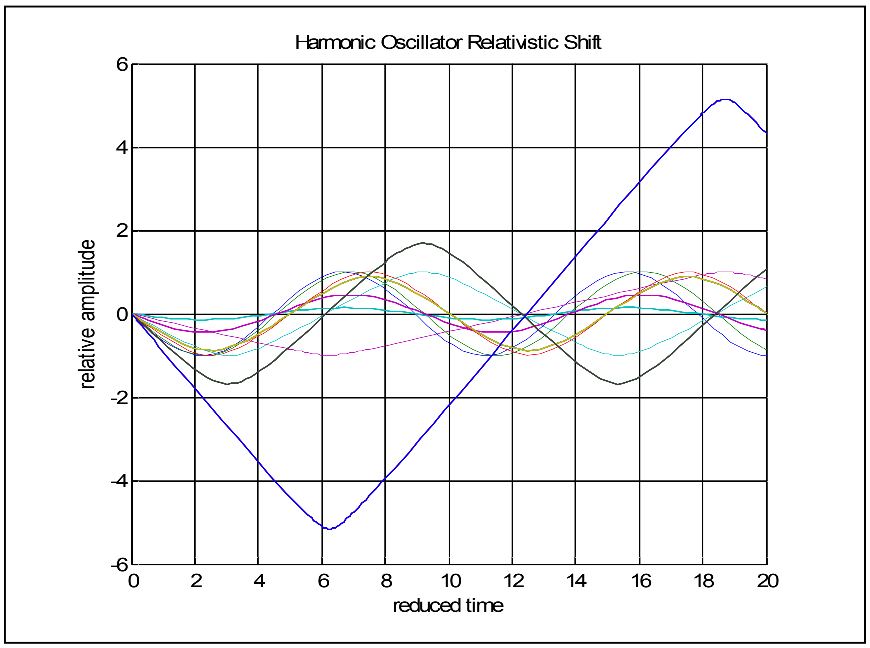
Apart from starting the problem for the mass at its maximum extension from the force center, the dimensionless displacement took values of 0.1, 0.3, 0.5, 0.7, and 0.9. These numbers also give the ratio of the potential energy to the particle mass energy according to the second member of (1.8). Thus the ratios are 0.005, 0.045, 0.125, 0.245 and 0.405.

Both the amplitude and period have relativistic shifts, but in this first set of plots, the relative excursions all have the same magnitude so that the nature of the relativistic period shifts would be more visible.



The color codes indicate the initial state expressed as the ratio of the potential energy to the particle mass energy.

As one expects, the oscillation period depends on the ratio of the maximum potential energy to the particle mass-energy. As the latter approaches the former, the motion period lengthens. Such behavior is expected since harmonic motion is a clock viewed from the laboratory frame and the period lengthening indicates the clock appears to be running more slowly – typical time dilation. But the relativity effect also shows up as a broadening of the transition as it picks up odd harmonics.



This plot is the same as above, but with the amplitude normalization eliminated. Clearly increasing anharmonicity is accompanied by increased displacement, ultimately leading to the failure of Hook's Law and subsequent fission.

This model might represent nuclear fission of a 'liquid drop' model along the reaction coordinate x. In that case the mass m is actually the mass defect Δm of the actinide. And the model shows that as the ratio of the maximum potential energy approaches the mass defect energy Δmc^2, the nucleus goes unstable. Now this model is admittedly a simplification since the spring constant k is not a constant in this circumstance but depends in a complicated way on the state of distension. In other words it is no longer a simple harmonic oscillator. More oscillatory motion occurs not just along the reaction coordinate but also in more complicated modes probably all described by a three-dimensional harmonic oscillator. Mode mixing will be an important feature of the fissioning process thus leading to several possible paths that result in disintegration of the nucleus.