$\frac{dy}{dt} = y^2 + t^2$ is a Riccati equation. We solve this by making the substitution; $y = -\frac{1}{f}\frac{df}{dt} \Longrightarrow$

$$\frac{dy}{dt} = \frac{1}{f^2}\left(\frac{df}{dt}\right)^2 - \frac{1}{f}\frac{d^2 f}{dt^2} = y^2 - \frac{1}{f}\frac{d^2 f}{dt^2}$$

$$\therefore y^2 - \frac{1}{f}\frac{d^2 f}{dt^2} = y^2 + t^2 \; OR \; \underline{\underline{\frac{d^2 f}{dt^2} + t^2 f = 0}} \qquad (1)$$

SOLUTION 1

The second order differential equation (1) above is the special Bessel equation whose solution is:

$$f = \sqrt{t}\left[AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{1}{4}}\left(\frac{t^2}{2}\right)\right]$$

Where A and B are arbitrary constants and J and Y are the Bessel functions of the first and second kind, respectively. Differentiating once we obtain:

$$\frac{df}{dt} = \frac{1}{2\sqrt{t}}\left[AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{1}{4}}\left(\frac{t^2}{2}\right)\right] + t\sqrt{t}\left\{A\left[\frac{1}{2t^2}J_{\frac{1}{4}}\left(\frac{t^2}{2}\right) - J_{\frac{5}{4}}\left(\frac{t^2}{2}\right)\right] + B\left[\frac{1}{2t^2}Y_{\frac{1}{4}}\left(\frac{t^2}{2}\right) - Y_{\frac{5}{4}}\left(\frac{t^2}{2}\right)\right]\right\}$$

$$= \frac{1}{2\sqrt{t}}\left[AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{1}{4}}\left(\frac{t^2}{2}\right)\right] + t\sqrt{t}\left\{\frac{1}{2t^2}\left[AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{1}{4}}\left(\frac{t^2}{2}\right)\right] - \left[AJ_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{5}{4}}\left(\frac{t^2}{2}\right)\right]\right\}$$

$$\frac{1}{f}\frac{df}{dt} = \frac{1}{2t} + t\left[\frac{1}{2t^2} - \frac{AJ_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{5}{4}}\left(\frac{t^2}{2}\right)}{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{1}{4}}\left(\frac{t^2}{2}\right)}\right]$$

$$= \frac{1}{t} - \frac{t\left[AJ_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{5}{4}}\left(\frac{t^2}{2}\right)\right]}{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{1}{4}}\left(\frac{t^2}{2}\right)}$$

$$Now\; y = -\frac{1}{f}\frac{df}{dt} = \frac{t\left[AJ_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{5}{4}}\left(\frac{t^2}{2}\right)\right]}{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{1}{4}}\left(\frac{t^2}{2}\right)} - \frac{1}{t}$$

Using the initial condition, $y(1) = 0$, we have;

$$0 = \frac{AJ_{\frac{5}{4}}\left(\frac{1}{2}\right) + BY_{\frac{5}{4}}\left(\frac{1}{2}\right)}{AJ_{\frac{1}{4}}\left(\frac{1}{2}\right) + BY_{\frac{1}{4}}\left(\frac{1}{2}\right)} - 1 \; OR \; \frac{AJ_{\frac{5}{4}}\left(\frac{1}{2}\right) + BY_{\frac{5}{4}}\left(\frac{1}{2}\right)}{AJ_{\frac{1}{4}}\left(\frac{1}{2}\right) + BY_{\frac{1}{4}}\left(\frac{1}{2}\right)} = 1$$

$$\Longrightarrow AJ_{\frac{1}{4}}\left(\frac{1}{2}\right) + BY_{\frac{1}{4}}\left(\frac{1}{2}\right) = AJ_{\frac{5}{4}}\left(\frac{1}{2}\right) + BY_{\frac{5}{4}}\left(\frac{1}{2}\right)$$

$$OR\; A\left[J_{\frac{1}{4}}\left(\frac{1}{2}\right) - J_{\frac{5}{4}}\left(\frac{1}{2}\right)\right] = B\left[Y_{\frac{5}{4}}\left(\frac{1}{2}\right) - Y_{\frac{1}{4}}\left(\frac{1}{2}\right)\right]$$

$$\Longrightarrow B = \left[\frac{J_{\frac{1}{4}}\left(\frac{1}{2}\right) - J_{\frac{5}{4}}\left(\frac{1}{2}\right)}{Y_{\frac{5}{4}}\left(\frac{1}{2}\right) - Y_{\frac{1}{4}}\left(\frac{1}{2}\right)}\right]A \; OR \; B = \kappa A \; with \; \kappa = \frac{J_{\frac{1}{4}}\left(\frac{1}{2}\right) - J_{\frac{5}{4}}\left(\frac{1}{2}\right)}{Y_{\frac{5}{4}}\left(\frac{1}{2}\right) - Y_{\frac{1}{4}}\left(\frac{1}{2}\right)}$$

$$Now\; y = \frac{At\left[J_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + \kappa Y_{\frac{5}{4}}\left(\frac{t^2}{2}\right)\right]}{A\left[J_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + \kappa Y_{\frac{1}{4}}\left(\frac{t^2}{2}\right)\right]} - \frac{1}{t} \; OR \; \underline{\underline{y = \frac{t\left[J_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + \kappa Y_{\frac{5}{4}}\left(\frac{t^2}{2}\right)\right]}{J_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + \kappa Y_{\frac{1}{4}}\left(\frac{t^2}{2}\right)} - \frac{1}{t}}} \qquad (2)$$

SOLUTION 2

If, however;

$$f = \sqrt{t}\left[AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)\right]$$

Then:

$$\frac{df}{dt} = \frac{1}{2\sqrt{t}}\left[AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)\right] + t\sqrt{t}\left\{A\left[\frac{1}{2t^2}J_{\frac{1}{4}}\left(\frac{t^2}{2}\right) - J_{\frac{5}{4}}\left(\frac{t^2}{2}\right)\right] + B\left[-\frac{1}{2t^2}J_{-\frac{1}{4}}\left(\frac{t^2}{2}\right) - J_{\frac{3}{4}}\left(\frac{t^2}{2}\right)\right]\right\}$$

$$= \frac{1}{2\sqrt{t}}\left[AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BY_{\frac{1}{4}}\left(\frac{t^2}{2}\right)\right] + t\sqrt{t}\left\{\frac{1}{2t^2}\left[AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) - BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)\right] - \left[AJ_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + BJ_{\frac{3}{4}}\left(\frac{t^2}{2}\right)\right]\right\}$$

$$\frac{1}{f}\frac{df}{dt} = \frac{1}{2t} + t\left\{\frac{1}{2t^2}\left[\frac{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) - BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)}{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)}\right] - \frac{AJ_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + BJ_{\frac{3}{4}}\left(\frac{t^2}{2}\right)}{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)}\right\}$$

$$= \frac{1}{2t}\left[\frac{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) - BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)}{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)} + 1\right] - \frac{t\left[AJ_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + BJ_{\frac{3}{4}}\left(\frac{t^2}{2}\right)\right]}{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)}$$

$$= \frac{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right)}{t\left[AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)\right]} - \frac{t\left[AJ_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + BJ_{\frac{3}{4}}\left(\frac{t^2}{2}\right)\right]}{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)}$$

$$y = \frac{t\left[AJ_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + BJ_{\frac{3}{4}}\left(\frac{t^2}{2}\right)\right]}{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)} - \frac{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right)}{t\left[AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)\right]}$$

Using the initial condition, $y(1) = 0$, we have:

$$0 = \frac{AJ_{\frac{5}{4}}\left(\frac{1}{2}\right) + BJ_{\frac{3}{4}}\left(\frac{1}{2}\right)}{AJ_{\frac{1}{4}}\left(\frac{1}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{1}{2}\right)} - \frac{AJ_{\frac{1}{4}}\left(\frac{1}{2}\right)}{AJ_{\frac{1}{4}}\left(\frac{1}{2}\right) + BJ_{-\frac{1}{4}}\left(\frac{1}{2}\right)} \Longrightarrow B = \frac{J_{\frac{1}{4}}\left(\frac{1}{2}\right) - J_{\frac{5}{4}}\left(\frac{1}{2}\right)}{J_{\frac{3}{4}}\left(\frac{1}{2}\right)}A$$

$$\therefore B = \kappa A \; where \; \kappa = \frac{J_{\frac{1}{4}}\left(\frac{1}{2}\right) - J_{\frac{5}{4}}\left(\frac{1}{2}\right)}{J_{\frac{3}{4}}\left(\frac{1}{2}\right)}$$

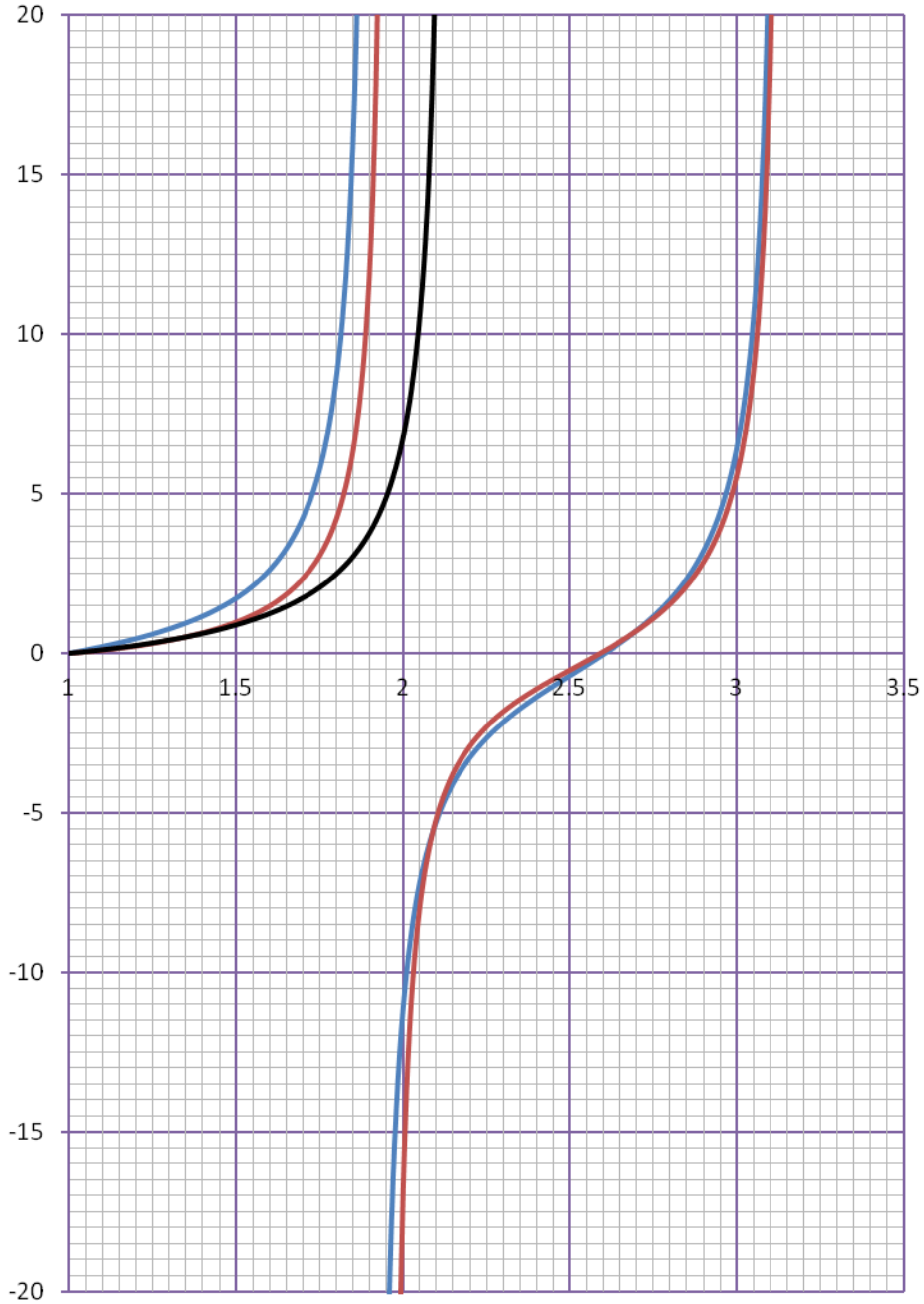
$$Now \; y = \frac{At\left[J_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + \kappa J_{\frac{3}{4}}\left(\frac{t^2}{2}\right)\right]}{A\left[J_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + \kappa J_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)\right]} - \frac{AJ_{\frac{1}{4}}\left(\frac{t^2}{2}\right)}{At\left[J_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + \kappa J_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)\right]}$$

$$OR \; y = \frac{t^2\left[J_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + \kappa J_{\frac{3}{4}}\left(\frac{t^2}{2}\right)\right] - J_{\frac{1}{4}}\left(\frac{t^2}{2}\right)}{t\left[J_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + \kappa J_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)\right]}$$

$$Now, J_{-\frac{1}{4}}\left(\frac{t^2}{2}\right) = \frac{3}{x^2}J_{\frac{3}{4}}\left(\frac{t^2}{2}\right) - J_{\frac{7}{4}}\left(\frac{t^2}{2}\right)$$

$$\therefore y = \frac{t^2\left[J_{\frac{5}{4}}\left(\frac{t^2}{2}\right) + \kappa J_{\frac{3}{4}}\left(\frac{t^2}{2}\right)\right] - J_{\frac{1}{4}}\left(\frac{t^2}{2}\right)}{t\left\{J_{\frac{1}{4}}\left(\frac{t^2}{2}\right) + \kappa\left[\frac{3}{x^2}J_{\frac{3}{4}}\left(\frac{t^2}{2}\right) - J_{\frac{7}{4}}\left(\frac{t^2}{2}\right)\right]\right\}} \tag{3}$$

We now compare solution (2) and (3) with the numerical solution (Runge-Kutta-Fehlberg with $h = 0.01$) in the graph below. The numerical solution for $h = 0.1$ is: $y(2) = 6.747927$. When the grid is refined, we obtain: $y(2) = 6.7096647805$. Both exact solutions will have hit discontinuity by the time and are unmatched.



Graph 1: Graph of $y(t)$ satisfying: $\frac{dy}{dt} = y^2 + t^2$. The numerical solution (——), the exact solution 1 (——) and the exact solution 2 (——).