ANALYSIS AND SYNTHESIS OF GENERAL PLANAR INTERFEROMETER ARRAYS

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ABSTRACT

This paper presents the first theory for general planar radio interferometers. Previously published results for the three antenna linear inteferometer are completely generalized to interferometers consisting of any number of antennas in arbitrary planar configurations. For any interferometer in this much wider class, the maximum likelihood bearing estimation algorithm can be derived, and its performance calculated using the results derived here. The antenna phase centers are viewed as generating a two dimensional lattice in the array plane. This lattice is the dual or reciprocal of the lattice in the direction cosine plane consisting of all direction cosine pairs which represent angles which are ambiguous with the array boresight. It is shown that the likelihood function for the unknown integer portions of the phase measurements can be reduced to an integer quadratic form which represents a generalized distance squared between an (N-2) dimensional projection of the phase measurement and the points of a lattice in phase space. The ambiguity resolution procedure is thus reduced to determining the closest lattice point to a given point.

INTRODUCTION

Phase interferometry is a commonly used technique for extracting information about the direction of a narrowband point radio source from the signals received by an array of antennas. If the received signal phase is measured without error at two different antennas, the difference between the two absolute phase measurements will be equal, in units of cycles, to the fractional part of the product of the antennas' separation in wavelengths and the cosine of the angle between the emitter direction and the line on which the two antennas lie. If the possible range of this direction cosine exceeds the reciprocal of the spacing in wavelengths, then the single relative phase measurement is ambiguous in the sense that it implies multiple values for the direction cosine. By using multiple antenna-pairs in a suitable geometric configuration, it is a simple matter to ensure that the mapping from the set of possible direction cosine pairs onto the set of relative phase measurement vectors is one-to-one and therefore invertible by a doa estimation algorithm. Determining a unique source direction from a given relative phase measurement vector involves, at least implicitly, determining the unmeasured integer portions of the relative phase measurements between all antennas pairs. This part of the direction of arrival estimation procedure is referred to as ambiguity resolution.

In practical applications errors arise due to several factors, most notably receiver noise, variations in the phase response of the antennas, and spatial variation in the electromagnetic field produced by the incident plane wave at each point in the array, due to a radome, for example. The latter two types of errors may be viewed as random variables if one considers the actual system realization as a sample from a large ensemble of identically manufactured systems. When all a priori knowledge about the various error sources is expressed in a joint conditional probability distribution function for the phase measurements given a true doa, the maximum-likelihood estimator can be described. This is done with variation in (1)-(4). (1) and (2) assume that the only error source is additive gaussian receiver noise, independent from channel to channel. For this assumption, a sufficient statistic consists of the detected I and Q amplitudes at each antenna's receiver. The likelihood function is multimodal, with local maxima corresponding closely to directions of arrival which imply exact phase measurements which differ by integer numbers of cycles. An exact maximum likelihood estimate for this case can thus be obtained by integrating the envelopes of the I and Q components of each antenna signal, and then maximizing the multimodal likelihood function for these measurements by some sort of iterative search which is exhaustive enough to always find the global maximum. Most practical estimation procedures achieve some simplification by assuming that the errors in the phase measurement due to receiver noise are gaussian random variables independent of the true doa. Kendall (1) derived a procedure for the three antenna linear array which involves computing maximum likelihood estimates for the absolute phase at each antenna's receiver, but then treats the resulting estimates as gaussian r.v.'s. The likelihood function for the direction of arrival then can be expressed as a quadratic form, which must be maximized over two variables, one discrete and one

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continuous. The two parts of the problem can be shown to separate into a one dimensional discrete problem (ambiguity resolution) and a continuous two dimensional least squares problem. An algorithm is described in (1), and its performance for different antenna spacings is derived. Behery and MacPhie present a method for N-antenna evenly spaced arrays using cross correlation to obtain a relative phase measurement. They also treat these estimates as gaussian random variables, and maximize the likelihood function by an iterative search.

In (3) and (4) the interferometers considered use crosscorrelation to obtain relative phase measurements, which are then approximated as gaussian random variables, with errors due to both noise and implementation factors. The results of these investigations follow (1) in the case of the three antenna linear interferometer. (3) also presents algorithms for linear interferometers with more than three antennas. In these cases the decision space for the ambiguity resolution decision is multidimensional, and the algorithms described are nonoptimal for the assumption of gaussian phase errors. Planar interferometers are discussed in (4) and (5). In (4) consideration is restricted to arrays of five antennas which consist of two orthogonal linear arrays with a common center element. The ambiguity resolution procedure described, which merely treats each three antenna sub-array independently, is not optimal if a common measurement is used in both parts of the algorithm. Hanson (5) considers arbitrary arrangements of antennas in a plane, and describes algorithms for resolving ambiguity in arrays with four and five elements. For planar interferometers of four and five antennas the dimension of the decision space in the ambiguity resolution process is one and two respectively, and Hanson's results are analogous to Goodwin's in that the four element algorithms are optimal, but those for the five element case are not.

The analysis presented in this paper is based on the same assumption made in (1)-(4), that the phase measurements may be approximated as a multivariate gaussian random vector. The maximum likelihood direction of arrival estimator is then completely described for any N-antenna planar interferometer. The ambiguity resolution part of the estimation algorithm is shown to be essentially to find the closest point in a lattice to a projection of the measurement vector. This interpretation of the problem has allowed the application of many results from the Geometry of Numbers, and has greatly simplified the synthesis of general planar interferometers.

PROBLEM FORMULATION

We consider here interferometer arrays of N+1 antennas in a plane. Phase errors arising from all sources are assumed to be treatable as gaussian random variables, independent from antenna to antenna, and independent of the incident wavefront bearing. We also assume for convenience that the phase measurements are made simultaneously. In this way we do not need to devote disscusion to the selection of a set of baselines from the N(N+1)/2 possibilities. Any N baselines which, if viewed as a graph, form a tree, will have equivalent performance when the phase measurements are optimally processed. Furthermore, no advantage in performance is obtained if additional phase comparisons are made.

The measurement vector can be written as:

$$\vec{\phi}=D^t\vec{u}+\vec{k}+\vec{\varepsilon} \tag{1}$$

where:

$D=(\vec{d}_1,\vec{d}_2,\ldots,\vec{d}_N)$

$\vec{d}_i$=vector baselines in wavelengths

$\vec{u}$=direction cosine vector of the emitter.

$\vec{k}$=integer vector uniquely determined by the requirement that:

$$0\le\phi_i<1$$

$\vec{\varepsilon}$=error in the phase measurement

$\vec{\varepsilon}$ is a zero mean gaussian random vector with covariance matrix R:

$$p(\vec{\varepsilon})=\frac{1}{(2\pi)^{N/2}|R|^{N/2}}\exp\{-\tfrac{1}{2}\vec{\varepsilon}^tR^{-1}\vec{\varepsilon}\}$$

A convenient likelihood function is the positive definite quadratic form:

$$L(\vec{u},\vec{k}|\vec{\phi})=(\vec{\phi}-D^t\vec{u}-\vec{k})\,R^{-1}(\vec{\phi}-D^t\vec{u}-\vec{k}) \tag{2}$$

which is a minimum for a given $\vec{\phi}$ when $\vec{u}$ and $\vec{k}$ are the optimum values. The problem adressed by this paper is that of finding practical estimation algorithms which will produce, for given $\vec{\phi}$, the values for $\vec{u}$ and $\vec{k}$ which minimize (2). The difficulty, of course, is the presence of the discrete variables k_i.

THE MAXIMUM LIKELIHOOD ESTIMATION PROCEDURE

The problem of finding efficient methods for minimizing (2) and for determining the performance of the optimal estimator can be treated very neatly by making use of the correspondence which exists between quadratic diophantine equations and lattices. A lattice may be defined as a discrete, finite dimensional set of vectors which is closed under addition and subtraction. There is an extensive literature on lattices and quadratic forms with integer variables; some references are (6)-(8).

As a first step, note that for two points $\vec{u}_1$, $\vec{u}_2$ in the direction cosine plane for which

$$\vec{d}_i\cdot\vec{u}_1=\vec{d}_i\cdot\vec{u}_2+z_i\ ,\quad i=1,2,\ldots,N$$

for some integers z_i,

$$\vec{\phi}(\vec{u}_1,\vec{\varepsilon})=\vec{\phi}(\vec{u}_2,\vec{\varepsilon})\ .$$

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Thus $\vec{u}_1$ and $\vec{u}_2$ cannot both be allowable direction cosine pairs if the interferometer is to determine a unique $\vec{u}$ given $\vec{\phi}$. The relationship between the allowable range of $\vec{u}$ and the array baseline goemetry can be stated succinctly with the aid of the following definition:

Two lattices Λ_1, Λ_2 in R^N are <u>dual</u> if:

(1) Λ_1, Λ_2 span the same subspace of R^N

(2) for any $\vec{x}$ in Λ_1, $\vec{y}$ in Λ_2,

$<\vec{x},\vec{y}>$ is an integer.

A further result that can easily be shown is that if $G = \{\vec{g}_i, \vec{g}_2, \ldots, \vec{g}_m\}$ generates Λ_1, then

$$G<G,G>^{-1}$$

will generate Λ_2. ($<G,G>_{ij} = <\vec{g}_i,\vec{g}_j>$.) We will use the notation:

$$\{G\} = \Lambda_1$$

to denote a lattice in terms of a generator matrix. Now the phase centers of the N+1 antennas generate a lattice in the array plane. The 2x2 matrix T is defined to be a generator for this array lattice. Thus $\vec{d}_i$ is in $\{T\}$ for $i=1,2,\ldots,N$. S is defined as the dual generator ($S^t = T^{-1}$). By rewriting eqn. (3),

$$\vec{d}_i \cdot (\vec{u}_1-\vec{u}_2) \blacksquare z_i \;, \; i=1,2,\ldots,N$$

we can see that the set of all points in the direction cosine plane which are ambiguous with a given point $\vec{u}$ is just the coset $\vec{u}+\{S\}$. The array phase measurements thus serve to determine $\vec{u}$ only modulo the lattice $\{S\}$. The array must thus be designed such that the a priori distribution of $\vec{u}$ is contained in some unit cell of the lattice $\{S\}$. Without loss of generality it can be assumed that:

(4) $\vec{u}=S\vec{\alpha}$ for some $0 \leq \alpha_i < 1$, $i=1,2$

i.e. $\vec{u}$ lies in the parallelogram whose edges are the columns of S.

Since the baselines $\vec{d}_i$ are in $\{T\}$ there is a 2 x N matrix of integers, P, such that:

(5) $D = TP$

Inserting (4) and (5) into (1),

(6) $\vec{\phi} = P^t\vec{\alpha}+\vec{k}+\vec{\varepsilon}$

and doing the same in (2),

(7) $L = (\vec{\phi}-P^t\vec{\alpha}-\vec{k})^t \, R^{-1}(\vec{\phi}-P^t\vec{\alpha}-\vec{k})$.

For fixed $\vec{k}$, the above quadratic form is minimized over $\vec{\alpha}$ for

(8) $\vec{\alpha} = (PR^{-1}P^t)^{-1}PR^{-1}(\vec{\phi}-\vec{k})$.

Inserting eqn. (8) into equn. (7) yields a likelihood function for $\vec{k}$ alone:

(9) $L(\vec{k}|\vec{\phi}) =$

$$(\vec{\phi}-\vec{k})^t(R^{-1}-R^{-1}P^t(PR^{-1}P^t)^{-1}PR^{-1})(\vec{\phi}-\vec{k}) \;.$$

By defining A to be an (N-2) x N matrix whose rows span the solution space to:

$$PR^{-1}\vec{x}=\vec{0} \qquad \vec{x} \text{ in } R^N \;,$$

(9) can be rewritten as:

(10) $L(\vec{k}|\vec{\phi})=$

$$(\vec{\phi}-\vec{k})^t(R^{-1}A^t(AR^{-1}A^t)^{-1}AR^{-1})(\vec{\phi}-\vec{k})$$

If $\vec{\psi}$ and $\vec{\zeta}$ are defined as the projections of $\vec{\phi}$ and $\vec{k}$ respectively onto the span of A^t,

(11) $\vec{\psi} = A^t(AR^{-1}A^t)^{-1}AR^{-1}\vec{\phi}$

(12) $\vec{\zeta} = A^t(AR^{-1}A^t)^{-1}AR^{-1}\vec{k}$

then:

(13) $L(\vec{k}|\vec{\phi}) = (\vec{\psi}-\vec{\zeta})^t \, R^{-1}(\vec{\psi}-\vec{\zeta})$.

Now since $\vec{k}$ is an integer vector, $\vec{\zeta}$ is in the lattice $\{A^t(AR^{-1}A^t)^{-1}AR^{-1}\}$, i.e. $\vec{\zeta}$ lies in the N-2 dimensional lattice spanned by the N columns of the projection matrix in the brackets. The likelihood function (13) is then the square of a generalized distance from the projection $\vec{\psi}$ of $\vec{\phi}$ to the lattice point $\vec{\zeta}$. Minimizing (13) over $\vec{k}$ can be shown to be equivalent to finding the closest lattice point to $\vec{\psi}$ if it can be shown that each lattice point is the projection of some

$$\vec{\phi}_0=P^t\vec{\alpha}_0+\vec{k}_0 \qquad \vec{\alpha}_0 \text{ in } [0,1)^2$$
$$\vec{k} \text{ in } Z^N.$$

In fact each lattice point in $\{A^t(AR^{-1}A^t)^{-1}AR^{-1}\}$ can be shown to equal some such $\vec{\phi}_0$.

It will be useful to further restrict the definition of A, so that:

(14) $AR^{-1} \equiv B$

is a matrix of <u>integers</u>. B^t then spans the lattice of solutions to

(15) $P\vec{x} = \vec{0} \qquad \vec{x} \text{ in } Z^N$.

Algorithms for solving such systems of linear diophantine equations can be found in many books on elementary number theory, for example (9). Since

$$AR^{-1}\vec{k} \blacksquare B\vec{k} \equiv \vec{m}$$

is now an integer vector, $A^t(AR^{-1}A^t)$ is a minimal dimension generator matrix for the lattice

$$\{A^t(AR^{-1}A^t)^{-1}AR^{-1}\} \;.$$

Now for any $\vec{m}_0$ in Z^{N-2}, the general solution to

$$B\vec{k}_0 = \vec{m}_0 \qquad \vec{k}_0 \text{ in } Z^N$$

is the sum of a particular solution $\vec{k}_1$ and the general solution to the homogeneous equation, which by the definition of B is just

$$\vec{k}_h = P^t\vec{n} \qquad \vec{n} \text{ in } Z^2 \;.$$

Thus

(16) $\vec{k}_0 = \vec{k}_1 + P^t\vec{n}$.

Since A^t, P^t span orthogonal subspaces of R^N, any vector in R^N can be decomposed into the sum of its projections onto the two subspaces.
Thus we can write:

(17)
$$\begin{aligned}\vec{\phi}_0 &= P^t\vec{\alpha}_0+\vec{k}_0 = \\ &= A^t(AR^{-1}A^t)^{-1}AR^{-1}\,\vec{\phi}_0 \\ &\quad + P^t(PR^{-1}P^t)^{-1}PR^{-1}\,\vec{\phi}_0 \\ &= A^t(AR^{-1}A^t)^{-1}\vec{m}_0 \\ &\quad + P^t(\vec{\alpha}_0 + \vec{n} + (PR^{-1}P^t)^{-1}PR^{-1}\vec{k}_1) \;.\end{aligned}$$

We can clearly choose $\vec{\alpha}_0$ in $[0,1)^2$, $\vec{n}$ in Z^2 to make the second term vanish, giving

(18) $\vec{\phi}_0 = P^t\vec{\alpha}_0+\vec{k}_0 = A^t(AR^{-1}A^t)^{-1}\vec{m}_0$.

For any lattice point then, (18) holds for some

$\vec{\alpha}_0$ in $[0,1)^2$, $\vec{k}_0$ in Z^N, and we can minimize equn. (10) by projecting $\vec{\phi}$ onto $\vec{\psi}$ as in (11), finding the closest point $\vec{\zeta}$ in $\{A^t(AR^{-1}A^t)^{-1}\}$, determining the corresponding value of $\vec{k}$, and using equns. (8) and (4) to get $\vec{u}$. In practice, since it is more efficient computationally to work with (N-2) dimensional vectors the procedure above is not followed exactly. A more efficient algorithm is:

1. Compute:
$$\vec{\psi}' = (AR^{-1}A^t)^{-\frac{1}{2}} \; \vec{\phi}$$
2. Find closest lattice point $\vec{\zeta}'$:
$$\vec{\zeta}' = (AR^{-1}A^t)^{-\frac{1}{2}} \; \vec{m}$$
i.e. minimize $|\psi' - \zeta'|^2$
3. Compute $\vec{k}$:
$$\vec{k} = \sum_{i=1}^{N-2} m_i \vec{k}_i$$
where $B\vec{k}_i = \vec{e}_i \qquad i=1,2,\ldots,N-2$
$$\vec{e}_i = (0,0,\ldots,0,1,0,\ldots,0)^t$$
↑ ith position
4. Use equns. (8) and (4) to get $\vec{u}$.

Step 2. is of course the most involved part algorithm. The set of all points $\vec{\psi}'$ which are closer to a given lattice point than to any other are referred to as the Voronoi region of the lattice. This region is the interior of a polytope, i.e. a convex region bounded by hyperplanes (10). Fast algorithms exists for some lattices, including A_n, the lattice for ambiguity resolution of the arrays in (2). These algorithms are derived in (11).

CONCLUSION

The maximum likelihood bearing estimation problem has been reduced to the problem of finding the closest point in a certain lattice to a projection of the phase measurement vector. This has resulted in the first computationally efficient implementation of the ML estimator, and has provided an invaluable insight into the problem of optimal synthesis of interferometer arrays. An example of a typical five and six antenna interferometer is illustrated in Figures 1 and 2.

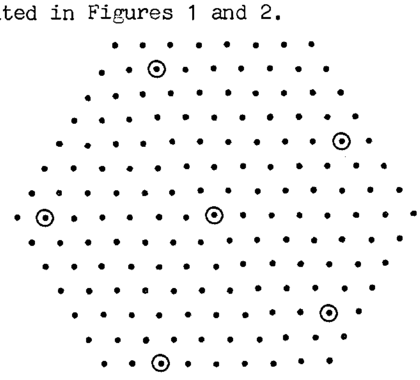

Figure 1. Six antenna array geometry and the array lattice generated by the antenna phase centers.

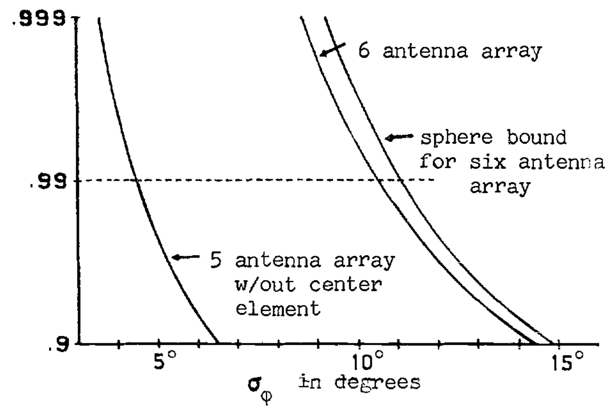

Figure 2. Probability of correctly resolving ambiguity versus the σ of the phase measurement error.

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