Theorem

$\forall x \in \mathbb{R}$, $\exists m \in \mathbb{Z}$ s.t. $m-1 \leq x < m$ (Note $\mathbb{Z}^+ = \{1, 2, 3, \dots\}$)

The archimedean property of $\mathbb{R}$ state that if $\bar{x}, \bar{y} \in \mathbb{R}$, $\bar{x} > 0 \Rightarrow \exists n \in \mathbb{Z}^+$ s.t. $n\bar{x} > \bar{y}$

Using this property with $\bar{x} = 1$ $\bar{y} = x$ we can find $m_1 \in \mathbb{Z}^+$ s.t. $m_1 \cdot 1 > x \Leftrightarrow x < m_1$

Using this property with $\bar{x} = 1$, $\bar{y} = -x$ we can find $m_2 \in \mathbb{Z}^+$ s.t. $m_2 \cdot 1 > -x \Leftrightarrow x > -m_2$

Hence we have found $m_1, m_2 \in \mathbb{Z}^+$ s.t. $\boxed{-m_2 < x < m_1}$. Note $-m_2, m_1 \in \mathbb{Z}$

let's define a set $T = \{t \in \mathbb{Z} : -m_2 \leq t \leq m_1\}$

Then $T \subset \mathbb{Z}$, T is not empty ($-m_2, m_1 \in T$) and T is finite

m_1 is the max of T, $-m_2$ is the min of T

We can define T as $\boxed{\{m_1, m_1 - 1, m_1 - 2, m_1 - 3, \dots -m_2 + 1, -m_2\}}$

Now we just need to show that $\exists m \in T$, s.t $m - 1 \leq x < m$

We argue by contradiction

Suppose $\nexists m \in T$ s.t $m - 1 \leq x < m$

We know that $x < m_1$, so we must have $x < m_1 \wedge x < m_1 - 1$

We also must have $x < m_1 - 1 \wedge x < m_1 - 2$ and so on until we get

$$\boxed{x < -m_2 + 1 \wedge x < -m_2}$$

But this is a contradiction since we have $x > -m_2$

Hence we conclude that $\exists m \in T$ s.t $m - 1 \leq x < m$

Since $m \in T \Rightarrow m \in \mathbb{Z}$, we have shown that

$$\exists m \in \mathbb{Z} \text{ s.t } m - 1 \leq x < m$$

$\square$