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CALCULUS I & LINEAR ALGEBRA I

MA111 Assignment 1

Semester I 2009

DUE : WEEK 7 THURSDAY 4PM
NOTE:
1. All questions are compulsory. 2. Show all relevant working.

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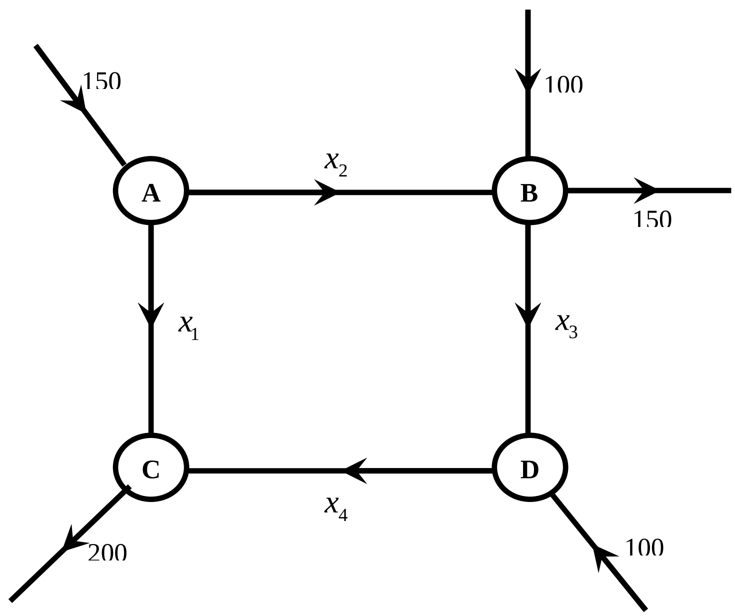
1. The figure below shows a network of one-way streets in Central Suva. On a particular day the traffic flow was measured at each of the intersections. The numbers in the figure represent the average number of vehicles entering and leaving every 10 minutes at each intersection.



(i) Set up a system for the traffic flow represented by x_i, $i = 1, \cdots, 4$.

(ii) Solve the system of linear equations from above using MATHEMATICA. Attach the computer printouts.

(iii) Discuss different patterns that will ensure congestion-free traffic flow, bearing in mind the possible restrictions that come into play.

(iv) Introduce at least one bypass/junction to the design and solve the new system. We will keep an eye for interesting scenarios. Attach computer printouts. **[20 marks]**

2. This question uses data from a hypothetical study; a study assessing the effectiveness of progressive exercise classes, in the management of lower back pain in primary care. The trial involved 500 patients aged 18-65 with a complaint of mechanical lower back pain lasting for at least four weeks or for 21 out of 28 days in the two months preceding randomisation[1] into the study. The outcome was the score on the Roland disability questionnaire (RDQ), a measure of disability caused by back pain rated on a scale from 0 (no disability) to 24 (severe disability), assessed at 3 months post-randomisation. You have been provided with the data from this study in an excel file: exercise.xls. The variables in this dataset are:

   idno Subject identification number
   exercise -1 = no exercise classes, 1 = exercise classes
   rdq_3m RDQ score at 3 months post-randomisation

Suppose we are required to use the Least Squares Regression Analysis (LSRA) to determine the effect of exercise on RDQ score at 3 months post-randomisation. In other words this LSRA will enable us to compare the RDQ score between the exercise and the no-exercise groups. We start by writing out the Linear Regression Model (LRM) for this 2-group (exercise group and no-exercise group) comparison in matrix form (i.e. matrix notation and the explicit matrices for each component of the model) by defining the following notations:

- Let Y_{ij} be the j^{th} observation (RDQ score at 3 months post-randomisation) in the i^{th} group; $i = 1 \text{ for exercise group},\ i = 2 \text{ for no-exercise group};\ j = 1,\ldots,n_i$. Y_{1j}'s represent observations in the exercise group, Y_{2j}'s represent observations in the no-exercise group, and n_i represents the number of observations in the i^{th} group.
- Let $X_{ij} = \begin{cases} 1 & if \text{ exercise} \\ -1 & if \text{ no-exercise} \end{cases}$
- Let e_{ij} denote the error term for the j^{th} observation in the i^{th} group

Thus, we obtain the matrix form for linear regression as:

$$\mathbf{Y} = \mathbf{XA} + \mathbf{E} \quad \text{where} \quad \begin{bmatrix} Y_{11} \\ \vdots \\ Y_{1n_1} \\ \hline Y_{21} \\ \vdots \\ Y_{1n_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \vdots & \vdots \\ 1 & 1 \\ \hline 1 & -1 \\ \vdots & \vdots \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} + \begin{bmatrix} e_{11} \\ \vdots \\ e_{1n_1} \\ \hline e_{21} \\ \vdots \\ e_{1n_2} \end{bmatrix}$$

[1] Randomisation means that each individual in the population (all individuals with lower back pain in the given area where the study took place) had an equal chance to be selected to participate in the study.

(i) *By hand*, construct the matrix $\mathbf{X'X}$ that is needed to obtain least-squares estimates for a_0 and a_1, and calculate its inverse using the fact that the inverse of a general matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{(ad-bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Again all your answers must be in general form using the notations introduced in part (i) above. [Hint: You may use $(a^2 - b^2) = (a+b)(a-b)$.]

(ii) Obtain general formulas for the least-squares estimate of a_0 and a_1 using the result from part (ii). [Hint: when calculating the matrix $\mathbf{X'Y}$ it may help to introduce a notation such as S_{yi}, for the sum of the Y values in the i^{th} group.]

(iii) Now use your results from part (iii) and the dataset provided to obtain the estimates of a_0 and a_1 (note that in this dataset $n_1 \neq n_2$). Also verify that $\overline{Y_1} = a_0 + a_1$, $\overline{Y_2} = a_0 + (-a_1)$, and $\overline{Y_2} - \overline{Y_1} = 2|a_1|$. [Note that $\overline{Y_1} = \text{mean of the exercise group}$, and $\overline{Y_2} = \text{mean of the no-exercise group}$.]

[15 marks]

3. If the Eigenvalues of

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

are $\lambda = 0$ and $\lambda = 1$, what are the possible values of *a* and *d*? **[5 marks]**

4. Find all eigenvalues and eigenvectors for the following system of differential equations.

$$y_1' = 5y_1 + 8/3\,y_2 - 2/3\,y_3\,,$$

$$y_2' = 2y_1 + 2/3\,y_2 + 4/3\,y_3$$

$$y_3' = -4y_1 - 4/3\,y_2 - 8/3\,y_3$$

[5 marks]

Show all the working. Check your answer using MATHEMATICA and provide computer printouts.

5. It is often important to transmit coded messages. For the code to be successful, it must be easily decoded by the intended recipient, but difficult to decode if intercepted by others. Many such codes use number theory and linear algebra. One effective method, due to Hill (1926), uses a large invertible matrix. We now outline this method but you should have at least seen this in your textbook.

   The sender and receiver agree on a choice of an invertible matrix M. For example,

$$M = \begin{bmatrix} -3 & 4 \\ -1 & 2 \end{bmatrix}.$$

   Now suppose we want to encode the message "ATTACK NOW". We first replace each letter with its position in the alphabet, that is, $a = 1, b = 2, \ldots, z = 26$. Spaces are not coded, but if there is an odd number of letters being coded, then we add a "0" at the end of the message. Thus the message "ATTACK NOW" is converted into the sequence of numbers 1, 20, 20, 1, 3, 11, 14, 15, 23, 0 which we then group as a sequence of column vectors

$$\begin{bmatrix} 1 \\ 20 \end{bmatrix}, \begin{bmatrix} 20 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 11 \end{bmatrix}, \begin{bmatrix} 14 \\ 15 \end{bmatrix}, \begin{bmatrix} 23 \\ 0 \end{bmatrix}.$$

   Multiplying each of these column vectors on the left by M gives

$$\begin{bmatrix} 77 \\ 39 \end{bmatrix}, \begin{bmatrix} -56 \\ -18 \end{bmatrix}, \begin{bmatrix} 35 \\ 19 \end{bmatrix}, \begin{bmatrix} 18 \\ 16 \end{bmatrix}, \begin{bmatrix} -69 \\ -23, \end{bmatrix} \tag{1}$$

   which results in the sequence

$$77,\ 39,\ -56,\ -18,\ 35,\ 19,\ 18,\ 16,\ -69,\ -23.$$

   This last sequence is the coded message that the sender transmits.

   To decode the message, the receiver computes

$$M^{-1} = \begin{bmatrix} -1 & 2 \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix},$$

   and then multiplies each of the column vectors in (1) by M^{-1} to get the original numbers, and hence the message. For example,

$$M^{-1} \begin{bmatrix} 77 \\ 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 20 \end{bmatrix} \text{ and } M^{-1} \begin{bmatrix} -56 \\ -18 \end{bmatrix} = \begin{bmatrix} 20 \\ 1 \end{bmatrix}.$$

   PROBLEM. The systems administrator in a rival organisation stores an employee's password by coding their initials together with their password. For example, if Geeta Edith Havili's password is glamis, then the administrator codes "GEHGLAMIS0".

   Suppose that you have located the coded initials and password of employee Fereti Osaiasi Singh as shown below.

$$33,\ 102,\ -193,\ 153,\ -207,\ 145,\ 100,\ 71,\ -60,\ 35$$

   Furthermore, suppose that an informant within the rival organisation tells you that the initials and password is coded using a 2×2 *symmetric* matrix. Can you determine Fereti's password? Are Fereti's password and, more importantly, the organisation's documents safe?

   [10 marks]