2210 volume formulas

Recall that the secret to finding area under a graph was to show that the derivative of the area function is the height function for the graph. Then the area function is the antiderivative of the height function, and this often lets us guess the area function from the height function. We can find many more formulas, such as volumes, by this method. All we have to do is figure out the derivative of the volume function, and then work backwards.

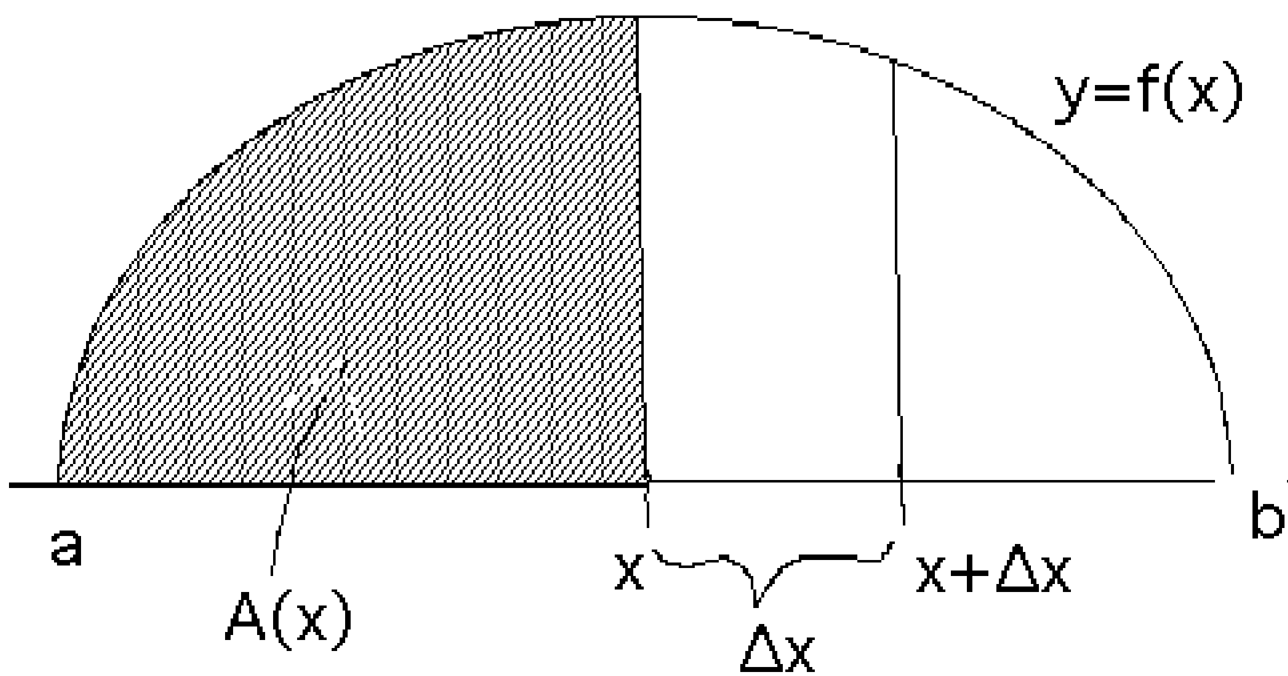
First remember how to calculate a derivative. If g is a function of x,

<u>the derivative of g at x</u>, written dg/dx, is:

the <u>limit</u> of $\Delta g/\Delta x$, as $\Delta x \to 0$,

where Δx = "change in x", and
$\Delta g = g(x+\Delta x)-g(x)$ = "change in g".

Let's review why the derivative of the area function for a graph is the height function, at least for continuous height functions.

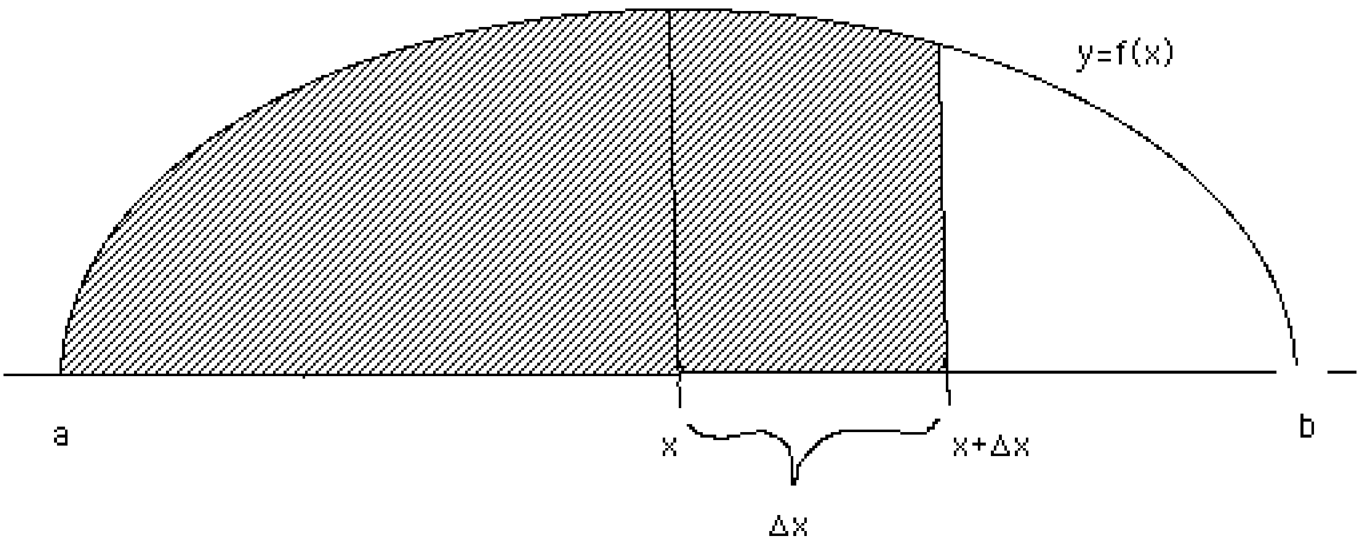


Define a function A(x) = "that part of the area under the graph of a continuous f, measured from a to x."

Then we claim the derivative of A at x, is simply f(x), the height of the graph, at x.

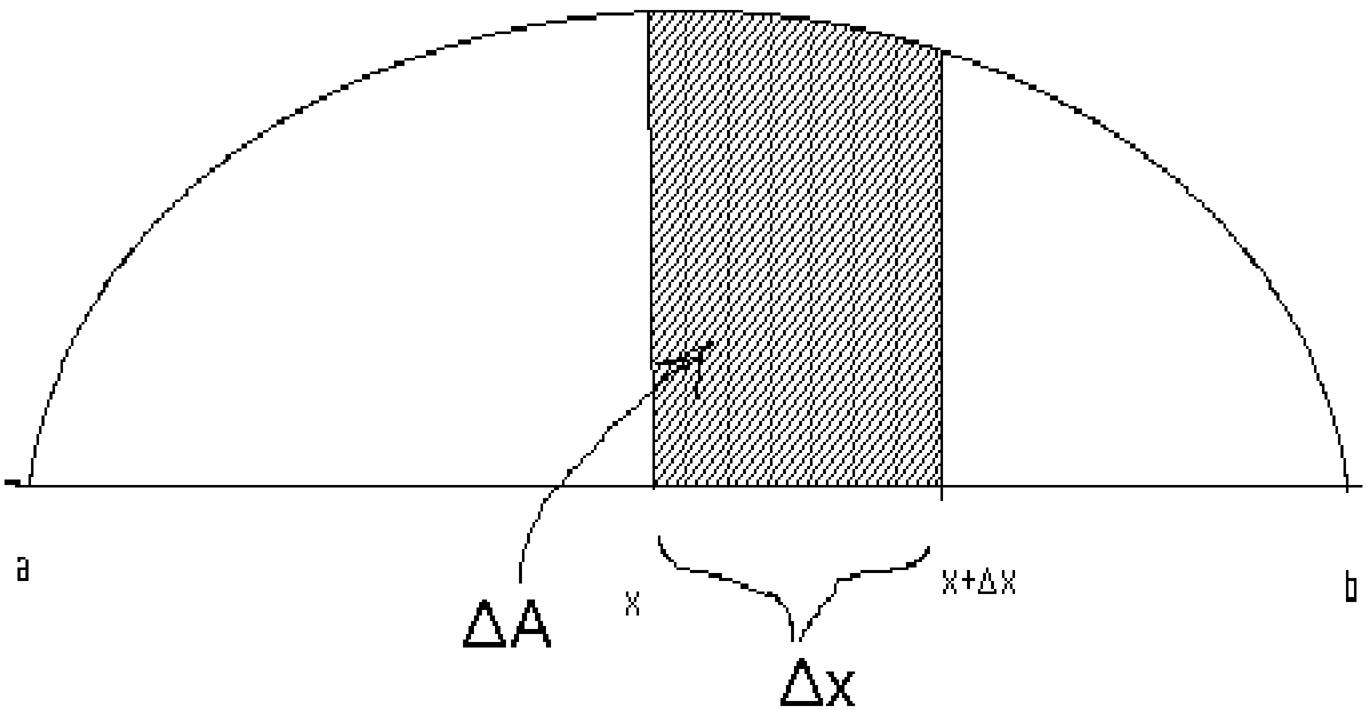
We must evaluate $\Delta A/\Delta x$, and take the limit as $\Delta x \to 0$. We see from the picture above that Δx = "change in x", is the (length of the) interval of the x axis between x and $x+\Delta x$.

We know also that ΔA = "change in A" = $A(x+\Delta x) - A(x)$, so next we picture these. We have shaded the area A(x) above, so next we shade $A(x+\Delta x)$ below.

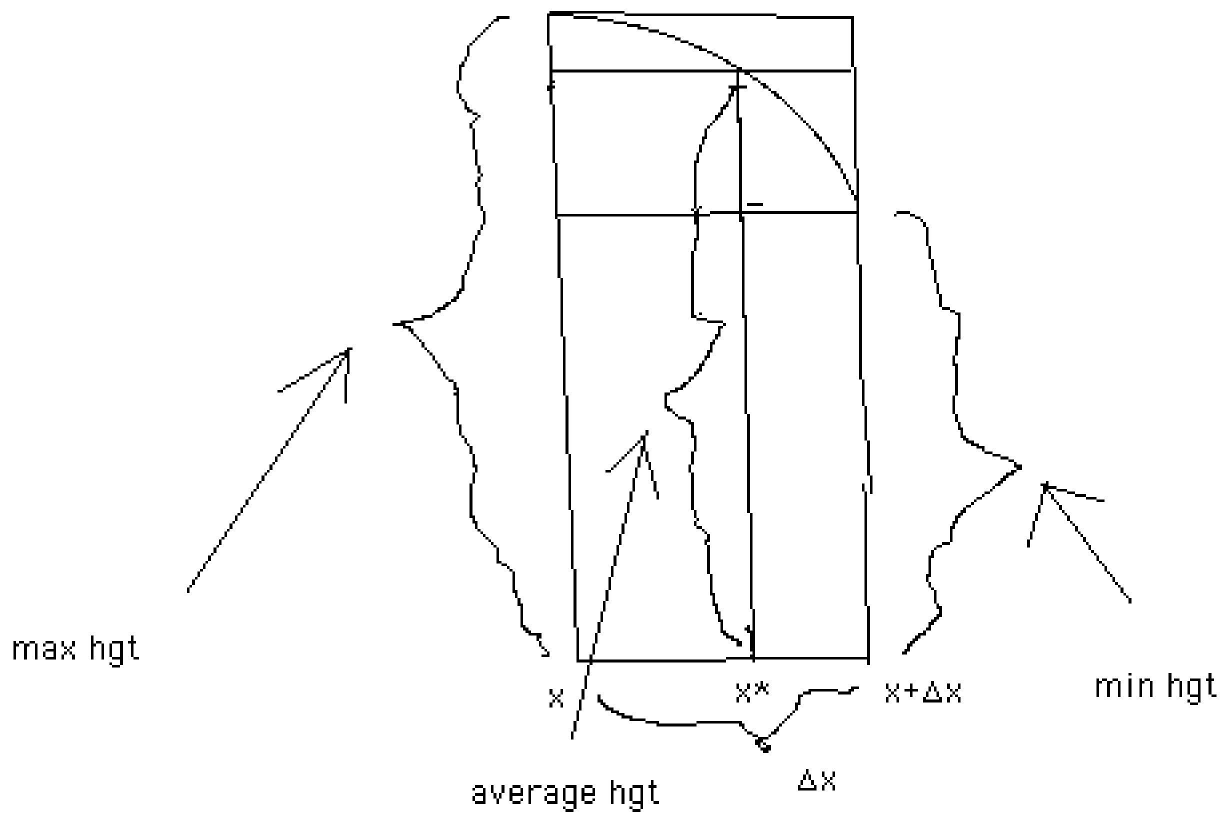


A(x+Δx) = shaded area = area between a and x+Δx.

To get ΔA, we subtract, i.e. $\Delta A = A(x+\Delta x) - A(x)$ = shaded area below.



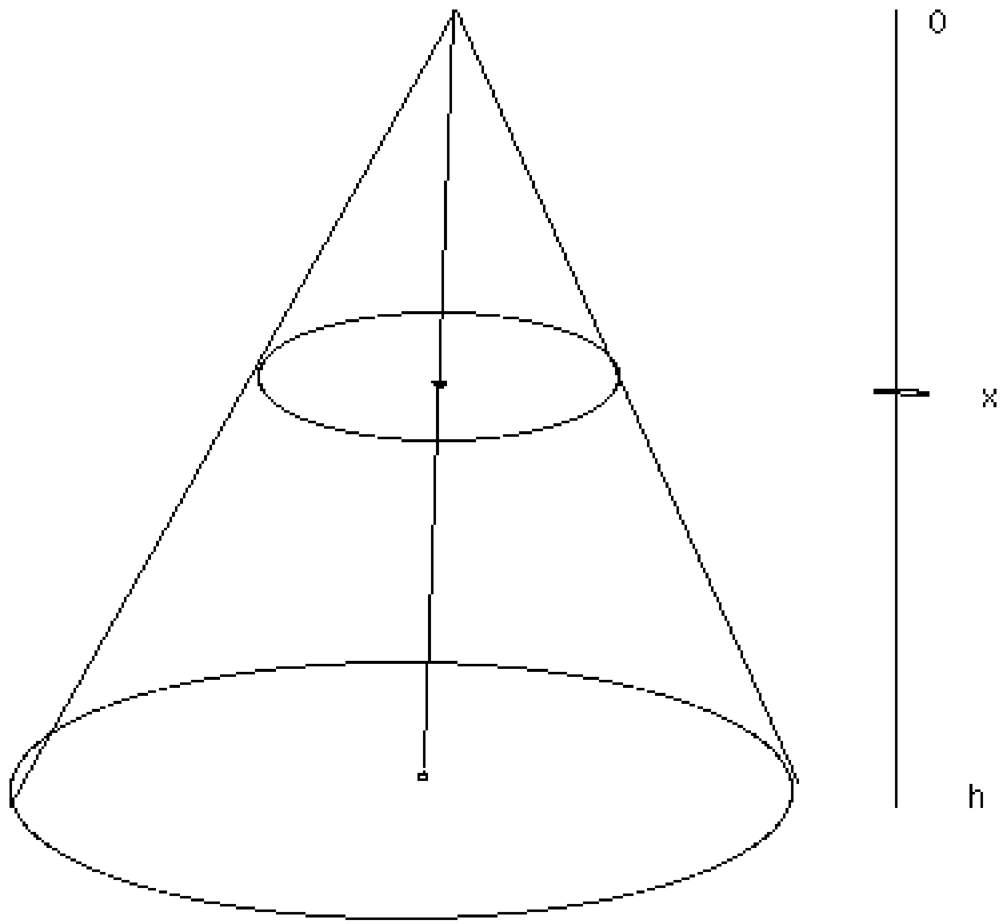
Now we have to divide $\Delta A/\Delta x$, and take the limit as $\Delta x \rightarrow 0$. I claim the quotient $\Delta A/\Delta x$, is the average height of the shaded region immediately above. I.e. when you multiply $\Delta A/\Delta x$ by Δx , you get the area ΔA of the shaded region. And Δx is the base of the shaded region. So ask yourself what number, when multiplied by the base of the region, gives the area? The answer is just the average height, shown below.
At least, in our picture it is some number between the height at x and the height at $x+\Delta x$.



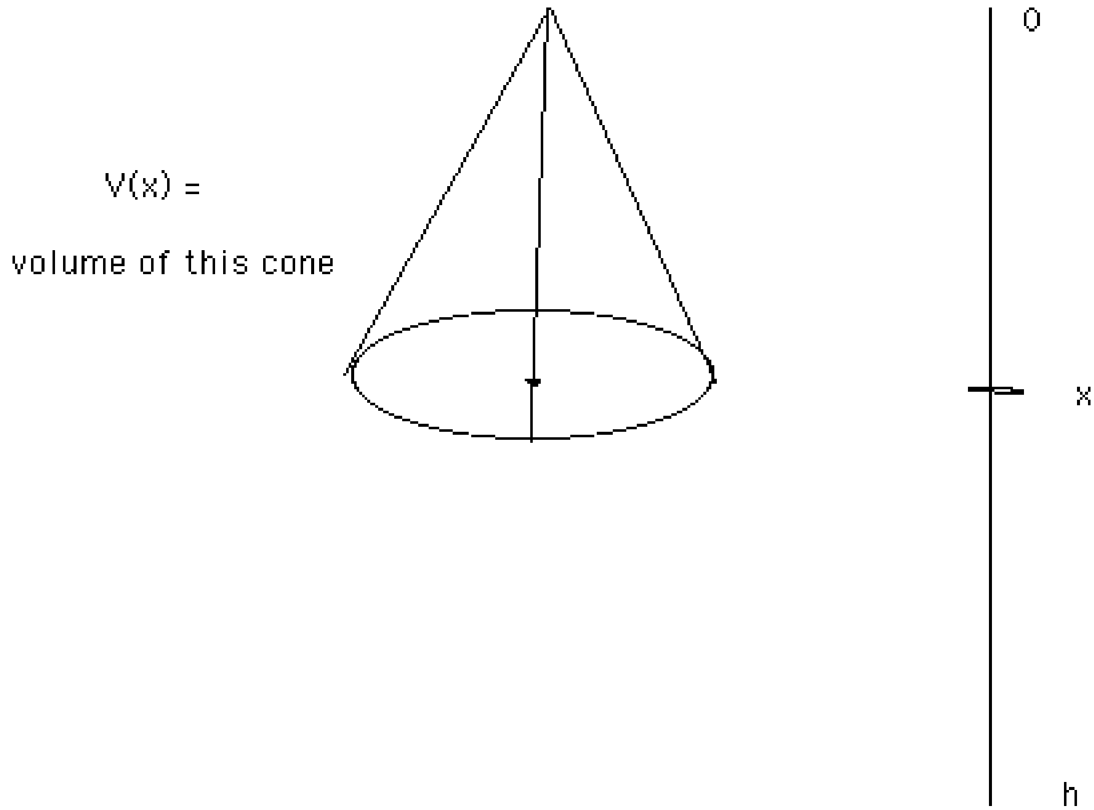
Now since $\Delta A/\Delta x$ is the average height of the graph between x and $x+\Delta x$, what is the derivative dA/dx = the limit of these average heights as $\Delta x \to 0$. For a continuous graph, this is just the actual height at x. I.e. for a continuous function, as the point x* where the average height is taken, approaches x, the height at x* approaches the height at x.
(Recall, f continuous at x <u>means</u> that, as x* approaches x, the limit of f(x*) is f(x).)

Application: If G is any antiderivative of f(x), the area function for f is G(x)-G(a).
proof: Two differentiable functions on an interval are equal if they have the same derivative, and are equal at one point. Now we just saw that $dA/dx = f(x)$, and by assumption also $d(G(x)-G(a))/dx = dG/dx = f(x)$ so we just need to compare A(x) and G(x)-G(a) at one point, such as a. But $A(a) = 0 = G(a)-G(a)$. Thus $A(x) = G(x)-G(a)$. I.e. if we can guesss any antiderivative of f, we can guess the area function for f.

Volume of a cone.
Next we try to find the volume formula of a cone, by the same method. Ie first try to find the derivative of the volume function. Look at the picture below of a cone.



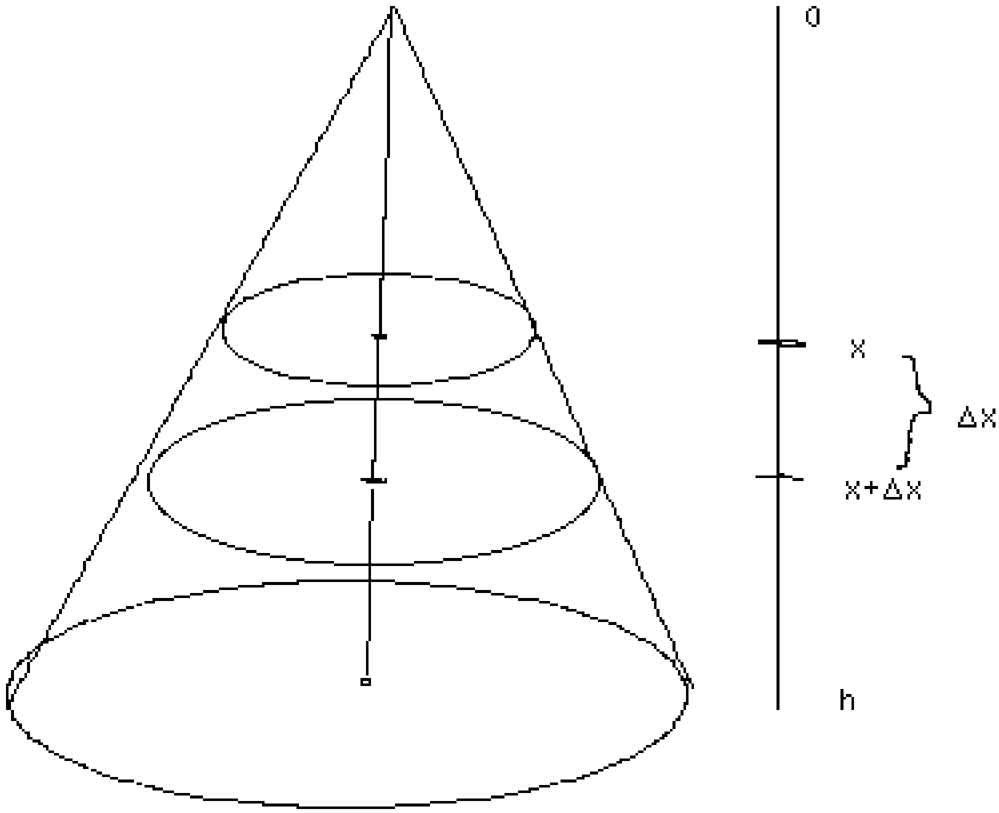
Now define a volume function V, where V(x) = the part of the volume from the top of the cone down as far as distance x from the top, i.e. V(x) is the volume of the small cone in the top part of the picture above. We show just this cone below.



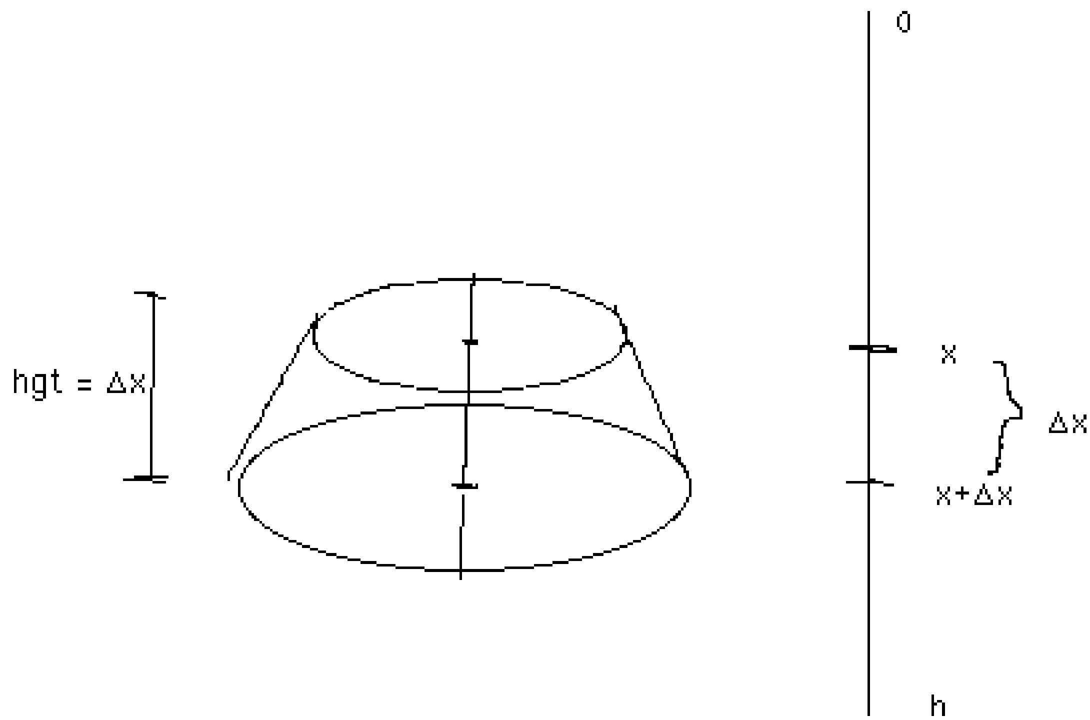
Then V(h), where h is the height of the cone, is the volume of the big cone. To figure out what the formula for V(x) is, we first ask what is the derivative dV/dx?

To calculate it we again start from the definition, and compute ΔV and Δx.

In the next picture we have shown two values of x, x and $x+\Delta x$, and the two corresponding cones, one whose base is a distance x from the top, and the second one with base a distance $x+\Delta x$ from the top. V(x) is the volume of the smaller cone, and $V(x+\Delta x)$ is the volume of the slightly larger one.



Then ΔV is the volume of the slab between the two cones, shown below.

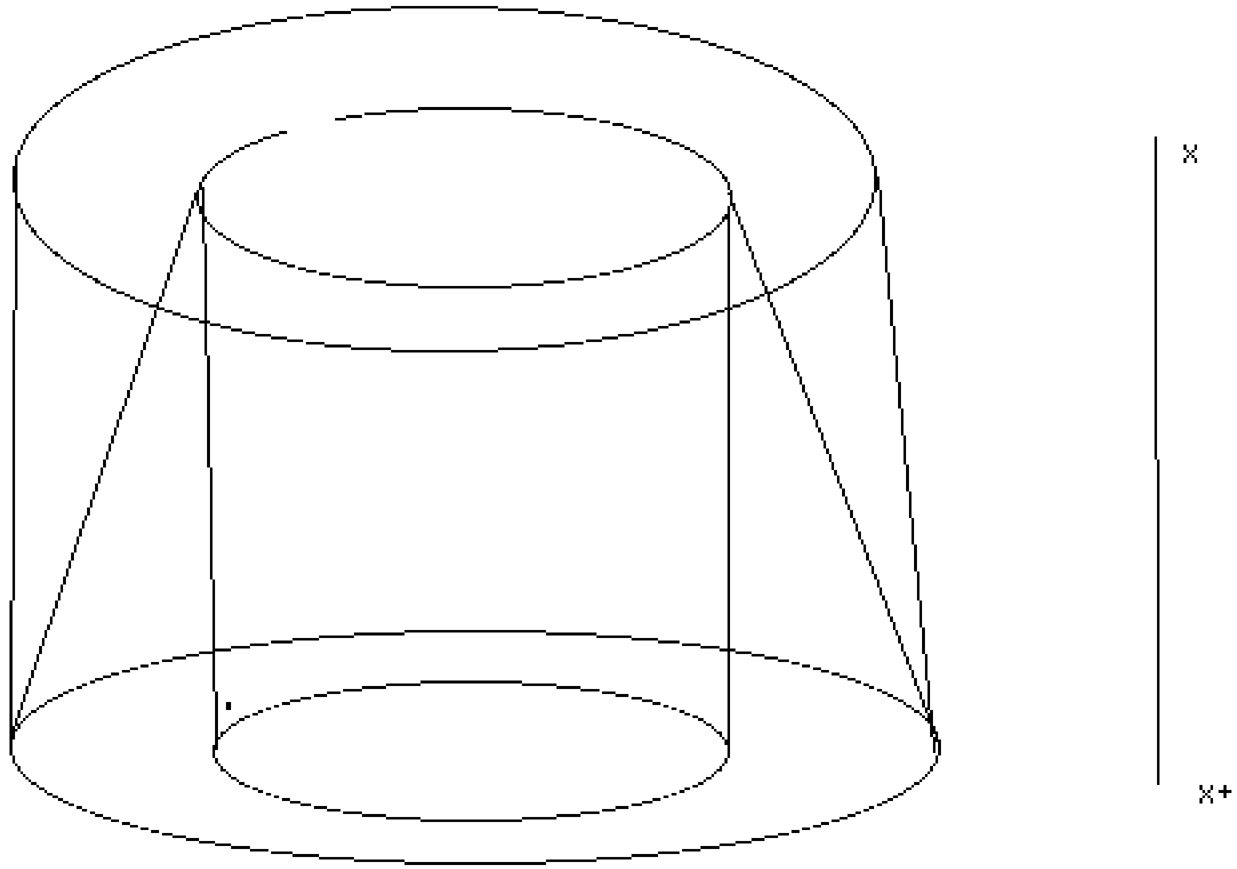


And we can see that Δx is the height of this slab.

Next we ask, what is the quotient $\Delta V/ \Delta x$? Since $\Delta V = (\Delta V/ \Delta x). \Delta x$, the quotient $(\Delta V/ \Delta x)$ must be some number which gives the volume of the slab when you multiply it by the height of the slab.

I claim this is the area of some circle between the top circle and the bottom circle. I.e. I claim that if you multiply the height of the slab by the area of the bottom circle, you get more than the volume of the slab, while if you multiply the area of the top circle by the height of the slab, you get less volume than that of the slab.

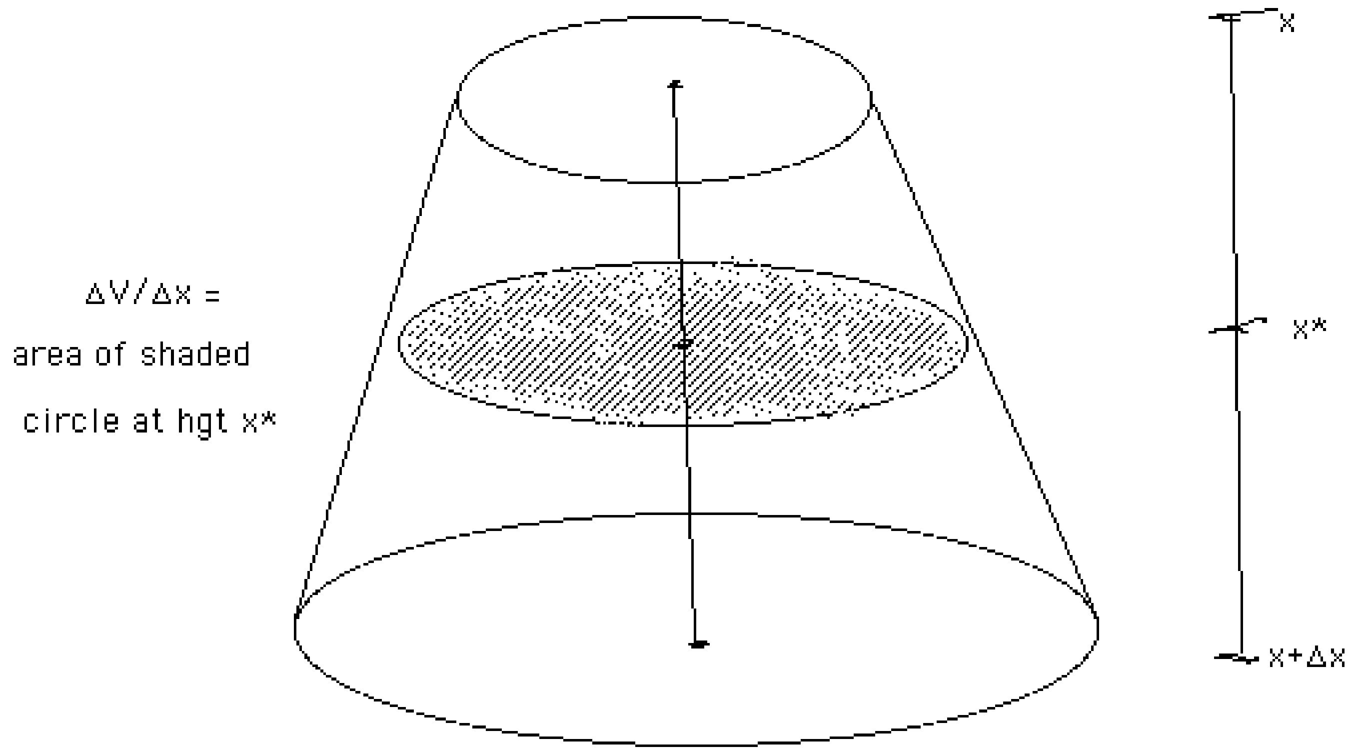
To see this, recall the volume of a <u>cylinder</u>, is obtained by multiplying the height by the area of the base. In our case the bottom circle is the abse of a cylinder that is larger than the slab, and the top circle is the base of a cylinder smaller than the slab, but all having the same height.



Volume of big cyclinder = Δx (area of large circle at bottom of conical slab)

Volume of small cylinder = Δx (area of small circle at top of conical slab) .

Thus volume of conical slab = Δx (area of some intermediate circle on conical slab).



Now to compute the derivative dV/dx, we ask what is the limit of ($\Delta V/\Delta x$) as $\Delta x \to 0$? I.e. what is the limiting value of the shaded area, as $x+\Delta x \to x$?

The answer seems to be the area of the circle at height x.
I.e. dV/dx = the area of the circle at height x.

I.e. the derivative of the volume function at x, seems to be the "area of a slice", at height x. So the derivative of a volume function seems to be an area function.

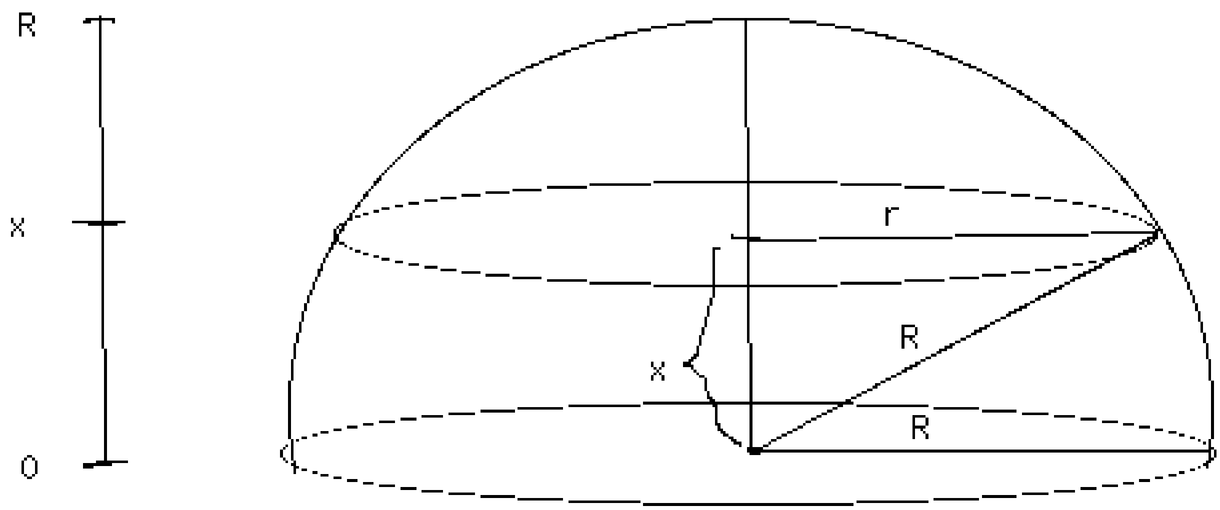
Lets apply this to find the volume of a cone and of a sphere. By our reasoning, we just need to find the corresponding area formulas, and then integrate.

Going back to the original cone, we ask for the area formula of the circle shown atdistance x from the top. Of course we just need its radius. If the radius of the bottom circle is R, then by similar triangles we have $x/h = r/R$, so $r = xR/h$. Thus the area function $A(x) = \pi r^2 = \pi(xR)^2/h^2 = (\pi R^2/h^2)x^2$. Thus an antiderivative is
$G(x) = (\pi R^2/h^2)x^3/3$.
Since the top has $x = 0$, the volume function is
$V(x) = G(x) - G(0) = G(x) = (\pi R^2/h^2)x^3/3$.

Thus the full volume is $V(h) = (\pi R^2/h^2)h^3/3 = \pi R^2 h/3$.

This is 1/3 the area of the base times the height, as expected.

Next we do the case of a hemisphere of radius R.
look at the picture below. If we define a volume formula V(x) to equal the part of the volume of the hemisphere up only as far as height x, then the derivative of this function, by reasoning exactly similar to that above, is the area of a slice, again circular, at height x. So we need a formula foir that area, and hence for the radius of that circle. Looking at the picture suggests using Pythagoras to get it.



I.e. by Pythagoras, we have $x^2 + r^2 = R^2$, so $r^2 = R^2 - x^2$. Thus the area we want is $\pi r^2 = \pi(R^2 - x^2)$. Thus $dV/dx = \pi(R^2-x^2) = \pi R^2 - \pi x^2$, so an antiderivative is
$G(x) = \pi R^2 x - \pi x^3/3$, and $G(0) = 0$, so again $V(x) = G(x) = \pi R^2 x - \pi x^3/3$. Thus the volume of the hemisphere is $V(R) = \pi R^3 - \pi R^3/3 = (2/3)\pi R^3$. Thus the volume of the full sphere is double this, or $(4/3)\pi R^3$, as discovered by Archimedes.

Exercise: Let a solid be formed by revolving the graph of $y = e^x$ around the x axis, betweem $x=0$ and $x = 4$. Define a volume function V(x) = the part of the volume inside this solid, between 0 and x.

Draw the picture of this solid.

What is the derivative of the volume function dV/dx?

Find the volume of the solid.

Exercise: Let a solid be formed with base on the plane $z = 1$, one unit above the x,y plane, and shaped something like the Eiffel tower, with each cross section (slice) parallel to the x,y plane and at height z, being a square of side $1/z$. if the tower reaches from $z=1$ up to $z = 12$, find the voluime of the tower.

(Draw the picture. Define a volume function to be $V(z)$ = that part of the volume between height 1 and height z. Find the derivative of this volume function. Find an antiderivative if possible, and find the volume.)

Do some problems in the book from the official syllabus.