Presented at the 7th Midwinter Convention of the American Institute of Electrical Engineers, New York, February 21, 1919.

ABNORMAL VOLTAGES WITHIN TRANSFORMERS

BY L. F. BLUME AND A. BOYAJIAN

ABSTRACT OF PAPER

Mathematical analysis is made of a rectangular wave impinging upon a transformer winding and quantitative values deduced of the resulting internal voltage stresses in terms of transformer constants. It is shown that the conclusions also apply in part to abrupt impulses and approximate idea is given of the reaction of a transformer to high frequencies. The difference between operating transformer with isolated and grounded neutral is shown. Energy losses are not considered in the mathematics although the manner in which the results are affected is pointed out. Finally, theoretical results are compared with impulse and high-frequency tests made in the laboratory.

I.—Introduction

To avoid breakdown of transformers resulting from abnormal voltages, the present practise is to either safeguard the transformer winding by external protective devices which are designed to prevent line disturbances from reaching the transformer, or to insulate the windings to withstand them. A third but less obvious method is so to design the transformer that voltage disturbances entering the winding will not result in excessive voltage concentration.

By showing in what way the transformer constants influence the voltage concentration this paper indicates how a transformer may be so designed as to mitigate the stresses occasioned by a given voltage disturbance.

II.—Conclusions

Theory and experiment both of which are fully described later in the paper confirm the following conclusions.

1. Transformers differ from transmission lines on account of the presence of coil-to-coil capacitance and mutual inductance.

2. Three kinds of disturbances are recognized:

A—abrupt impulses: B—rectangular waves with long tails: C—high-frequency voltages sustained and damped.

A. Abrupt Impulses

3. The presence of coil capacitance causes the transformer to respond as a capacitance and not as an inductance to abrupt impulses and towards all frequencies* above its lower natural frequencies of oscillations.

4. The voltage distribution within the transformer windings for abrupt impulses and for high frequencies depends upon the factor,

$$\alpha = \sqrt{\frac{C_g}{C_s}}$$

where C_g is the total capacitance of winding to ground and C_s is the total coil capacitance from one end of the coil stack to the other.

5. The greater the value of alpha, the greater the concentration of voltage at the line end and the smaller the voltage in the interior of the winding. The maximum volts per unit length of coil stack at the line end is equal to alpha times the value of the voltage corresponding to uniform distribution. In high-voltage power transformers of disk high-voltage, and concentric-barrel low-voltage windings, the value of alpha varies from 5 to 30.

6. The more abrupt the impressed impulses, or the higher the impressed frequency, the greater is the concentration of voltage at the line ends. The maximum voltage expressed in the preceding paragraph corresponds to a perpendicular wave front or infinite frequency impressed.

7. If the turns are uniformly distributed throughout the windings, alpha is also the factor giving the concentration of the volts per turn at the line end; but on account of extra turn insulation, the usual practise is to considerably reduce the number of turns on the end coils.

As the voltage concentration in the end coils depends upon the value of series and ground capacitances and is independent of the number of turns, it follows that the volts per turn on the end turns is inversely as the number of turns per coil. For this reason it is evident that extra turn insulation will be effective only when it is so designed that the increase in dielectric strength is greater than the increase in voltage concentration.

8. For abrupt impulses and for very high-frequency

*See paragraph 15.

oscillations it is immaterial whether the neutral is isolated or grounded.

B. Rectangular Waves with Long Tails

9. If the impulse has a long tail, the phenomena within the transformer will be initially as described in paragraphs 1 to 5 above, followed by a series of damped oscillations, the amplitudes of these oscillations depending not only on the amplitude of the impressed wave but also on the value of alpha, and on whether the neutral is isolated or grounded.

10. In addition to being proportional to the value of the impressed wave the amplitude of oscillation for a given harmonic is smaller the higher the harmonic and the smaller the value of alpha.

11. There is no simple relation between wave length and frequency and on that account waves cannot penetrate into a transformer winding without distortion.

12. As these oscillations occur simultaneously within the transformer windings, the resultant voltage distribution at any instant is obtained by re-combining the harmonics at that instant.

13. For isolated neutral and for alpha equal to 10, a maximum of 2.8 times impressed volts exists from neutral to ground and the volts per turn in the interior of the windings is approximately four times normal.

14. For grounded neutral and for alpha equal to 10, a maximum of 1.5 times impressed volts exists from the windings to ground, and the volts per turn in the interior of the windings are increased to four times normal and at the neutral to 7.5 times normal.

C. High Frequency

15. When the applied high frequency is greater than the natural frequency of the transformer, the reaction of the transformer is similar to its reaction towards abrupt impulses, and paragraphs 3 to 8 apply.

16. Theoretically, by impressing a sustained high-frequency voltage across transformer terminals at a natural oscillating frequency of the windings, the transformer can be made to resonate and indefinite voltages built up limited only by internal losses. However, if the transformer is excited by a transmission line, the voltage built up by resonance is also

limited by the relative values of surge impedance of transformer and line.

17. However in the high frequency tests here reported extreme high voltages were not observed. This is probably due to the following reasons:

(a) The impressed high-frequency was highly damped.
(b) The presence of internal capacitances shunting the windings decreases the rapidity with which the voltage is accumulated.
(c) Internal losses within the transformer windings.

18. The effects of (b) and (c) are greater the higher the frequency of oscillation. For this reason high internal voltages are built up, only at the lower frequencies.

III.—Description of Phenomena

The theory of electrical waves in transmission lines cannot be directly applied to transformers due to the fact that a transformer, unlike a transmission line, possesses capacitance between coils and mutual inductance.

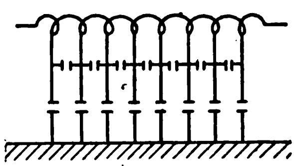


Fig. 1

Assuming that the inductance and capacitances are uniformally distributed along the winding as diagrammatically represented in Fig. 1, mathematical analysis can readily be applied, but if the constants are bunched as is the case for transformers in which the high and low voltage windings are interlaced, the problem becomes very complicated.

A good example of a concentric winding transformer to which the conclusions of this report are applicable is given in Fig. 4. This is a three-phase 2500-kv-a. transformer, the 100,000-volt winding of which is Y connected and made up of three stacks of disk coils. It is necessary to assume that the capacitances between turns and layers are very large compared with capacitances between coils and ground to make Fig. 1 applicable.

The purpose of this paper is to show what takes place within the transformer windings under the impact of rectangular waves. Rectangular waves are chosen rather than periodic waves on account of the greater simplicity of the problem, and

*See *Abnormal Voltages on Transformers*, by J. M. Weed, A. I. E. E. Trans. 1915, p. 1621.

waves of long duration (long compared with the time of a complete oscillation within the transformer) are assumed because the resulting voltages are greater than with the very short waves or impulses.

In general there will be two cases requiring separate treatment depending upon whether the neutral is isolated or grounded.

Although the phenomena with which we are dealing involves in its entirety only a very small fraction of a second, it is nevertheless convenient to divide it into three distinct time periods, each requiring separate treatment. These will be designated; 1, initial period; 2, transient period; 3, final period.*

The initial period is the period from the beginning of the disturbance to the time of maximum applied voltage. The time involved is so small that there is no appreciable growth of magnetic flux and therefore the phenomena may be considered entirely electrostatic without appreciable error. This is followed by a transient period during which there is a continual interchange of energy between the electric and magnetic fields. The energy interchange usually takes the form of highly damped high-frequency voltage and current oscillations. When the oscillations become negligible the final period has been reached in which the winding has again assumed a permanent condition of equilibrium.

Transformer Reacting as a Condenser

Due to the combination of coil and ground capacitances, the transformer, during the initial period, reacts as a concentrated capacitance. The condensers are rapidly charged and the voltage across them rises to nearly double the voltage of the incoming wave. The time required for charging is so very small that it is safe to assume that twice the voltage of the incoming wave is applied to the terminals of the transformer windings before an appreciable inductive current can be established in the windings. The equivalent or effective capacitance of the transformer, while being charged, is approximately given by equation (**21**),

$$C_{eff} = \sqrt{C_g \times C_s} \qquad \textbf{(21)}$$

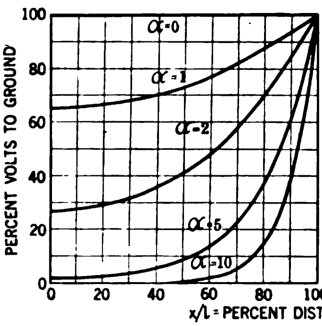
where C_g is the total capacitance of the surface of transformer

*See paper by Karl Willy Wagner entitled "Progress of Waves in Windings." *Elektrotechnik und Maschinenbau.*" Feb. 21, and 28th, 1915.

winding to ground, C_s is the capacitance between the two ends of the winding. The calculation of the latter capacitance is simplified by assuming that the ends of the winding are electrically connected to clamping plates or end rings. C_s is then defined as the capacitance between these plates.

Initial Voltage Distribution

The voltage distribution throughout the transformer winding when the condensers are charged is given by formula (**11b**) for isolated neutral, and formula (**13b**) for grounded neutral.



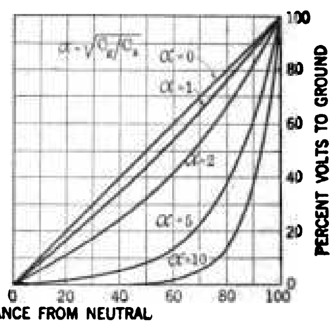


FIG. 2—NEUTRAL ISOLATED FIG. 3—NEUTRAL GROUNDED

$$e = E_0 \frac{\cosh \frac{\alpha x}{l}}{\cosh \alpha} \qquad \textbf{(11b)}$$

$$e = E_0 \frac{\sinh \frac{\alpha x}{l}}{\sinh \alpha} \qquad \textbf{(13b)}$$

where E_0 is value of voltage to ground on the transformer terminals. This will be either twice the value of the incoming wave, or, if lightning arresters are assumed, the maximum voltage permitted by the protective device.

The value of alpha is the square root of the ratio of the ground capacitance (C_g) to coil capacitance (C_s).

Plotting the ratio e/E_0 as ordinate and the ratio x/l as abscissa gives the system of curves shown in Figs. 2 and 3. For alpha equals zero, that is, ground capacitance zero, the curve is a straight line (uniform voltage distribution). The greater the value of alpha the less uniform becomes the voltage distribution and the more is the voltage concentrated at the end of the winding. This is the reason for the modern practise of putting extra insulation on the coils and turns at the line ends.

It should be noted that these curves are based on windings in which the turns are uniformly distributed along the coil stack. On account of the space required by the extra end turn insulation it follows that there are fewer turns per unit length of coil stack at the ends of the windings. The effect of this is to cause a still greater concentration of voltage at the ends.

Final Value of Voltage Distribution

(a) Eventually, in case of isolated neutral, the voltage of the winding will be brought to a potential of twice the original value of the wave with zero current flowing.

(b) For the case of grounded neutral the final condition will be a straight line with neutral at ground potential and line at twice the voltage of the original wave, with a current in the winding increasing at a constant rate, (assuming no resistance).

Transient Voltage Distribution

The transient by which the voltage distribution changes from the initial to the final value can be analyzed into a complex series of damped oscillations or standing waves at various frequencies, wave lengths and damping factors.

The maximum values of these standing waves are readily obtained by analyzing the initial voltage distribution as given in curves, Fig. 2, into space harmonics with respect to the final voltage distribution as axis of reference. Thus we have for $\alpha = 10$ the following initial values of the standing waves.

NEUTRAL ISOLATED		NEUTRAL GROUNDED.	
Wave length on winding n	Amplitude in per-cent of line to ground voltage E_n/E_0	Wave length on winding. n	Amplitude in per cent of line to ground voltage e/E_0
1/4	125%	1/2	58%
3/4	34%	3/2	23%
5/4	15%	5/2	6%
7/4	7%	7/2	4%
9/4	3%	9/2	3%

These oscillations at various frequencies occur simultaneously within the transformer windings and the complex wave at a given time is obtained by recombining the harmonics at that time (Fig. 5).

The equation of any standing wave in a transformer winding. is

$$e_x = E_n \operatorname{Sin}\left(2\,\pi\, n\,\frac{x}{l}\right) \cos w_n t$$

or (33d)

$$e_x = E_n \operatorname{Cos}\left(2\,\pi\, n\,\frac{x}{l}\right) \cos w_n t$$

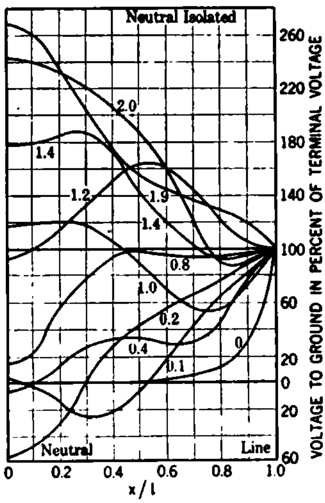


Fig. 5—Voltage Distribution within Transformer Windings at Various Times after Impact of Rectangular Wave

where n is the number of wave lengths per harmonic along the stack. For the fundamental, n is ½ when neutral is grounded, and ¼ when neutral is isolated. The frequency of oscillation is given by the equation:

$$f = \frac{2\,\pi\, n^2}{\sqrt{L\,(C_g + 4\,\pi^2\, n^2\, C_s)}} \tag{35e}$$

These equations are particular solutions of the general differential equations and are derived under the heading of "Mathematical Development."

An inspection of formula (35e) shows that the natural frequencies are a function of the wave lengths. For $\alpha = 10$ the variation of wave lengths with frequency is as follows:

NEUTRAL ISOLATED		NEUTRAL GROUNDED	
Wave length	Relative * frequencies	Wave length	Relative * frequencies
n		n	
1/4	1	1/2	4
3/4	8	2/2	14
5/4	20	3/2	28
7/4	33	4/2	42
9/3	47	5/2	56

*These figures are not accurate due to the variation of inductance with wave length as discussed on another page.

This peculiar relation between frequency and wave length is due to the presence of coil capacitance and mutual inductance, and causes a continual change in the form of the complex wave from instant to instant. This is in marked contrast to the phenomena which take place in the transmission line, where a complex wave of any shape will be propagated without distortion, except for the effect of resistance.

Maximum Voltage to Ground and Maximum Volts Per Turn

The considerations of greatest interest in connection with these curves are the maximum voltages to ground and the maximum volts per turn at various points within windings. These are plotted in Figs. 6 and 7. Of course the maximum values of voltages at the various points do not occur at the same instant.

An inspection of these curves shows that, for the case of isolated neutral, the maximum voltage to ground occurs at the neutral and is equal to 2.8 times the voltage from line to ground and for grounded neutral the maximum volts to ground is 1.5 times line to ground voltage. The volts per turn for isolated neutral is 10 times normal at line end, 4 times normal in the major portion of winding, and zero at the neutral. With neutral grounded it is 10 times normal at line end 5 times normal in the major portion of the winding and 7.5 times normal at the neutral.

SURGE IMPEDANCE

In order to have a convenient means of calculating the maximum value of current associated with each oscillation the ratio of terminal volts to maximum current is given by

$$Z = \frac{E}{I} = \frac{1}{2\pi n}\sqrt{\frac{L}{C_g + 4\pi^2 n^2 C_s}} \qquad (39b)$$

The value of surge impedance calculated for the transformer shown in Fig. 4 is 60,000 for the fundamental, 29,000 for the

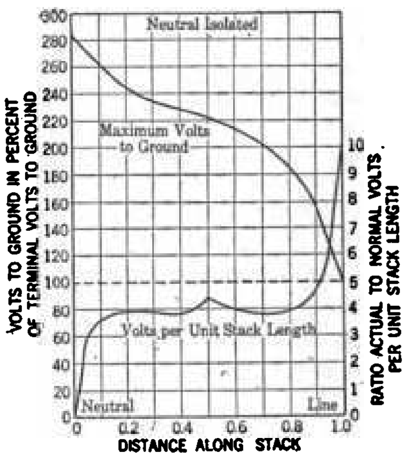


FIG. 6—VOLTAGE DISTRIBUTION WITHIN TRANSFORMER WINDINGS MAXIMUM VALUES AFTER IMPACT OF RECTANGULAR WAVES

third harmonic, 17,000 for the fifth harmonic and 12,000 for the seventh harmonic.

In the analysis just given it has been tacitly assumed that the voltage E applied across the terminals of the transformer winding is maintained throughout the oscillations. Strictly speaking, this is not true, because during the time when the voltage to ground is increasing in the interior of the winding, the transmission line is delivering current to the transformer windings, and during the time when this voltage to ground is decreasing, the transmission line is receiving current from the transformer windings, or in other words, there is a continual interchange of energy between the transmission line and the winding throughout the oscillation, and as the ratio of voltage to current associated in the rectangular wave in the transmission line is a constant value, this interchange of energy cannot take place without modifying the line voltage. However, calculations of equivalent surge impedance show that in all practical transformers the amount of energy absorbed and delivered by the transformer windings is very small compared with the energy residing in the rectangular wave, and that, therefore, this fluctuation in voltage is negligible in amount.

Again it may at first appear that the axis of oscillation taken in the case of a grounded-neutral transformer, cannot be correct

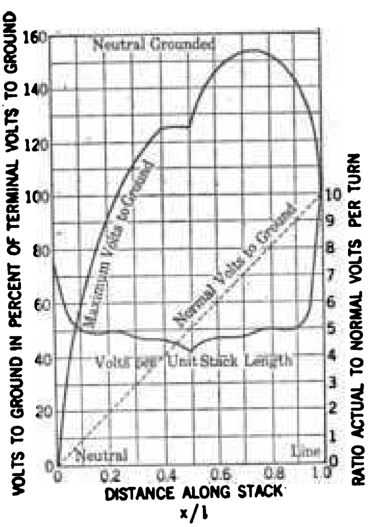


FIG. 7—VOLTAGE DISTRIBUTION WITHIN TRANSFORMER WINDINGS
MAXIMUM VALUES AFTER IMPACT OF RECTANGULAR WAVE

because the final condition for the case of grounded neutral is zero voltage throughout the windings owing to the fact that the windings become a dead short circuit. This is again theoretically true but it has no bearing on the problem because in all practical cases the time taken for the transformer windings to degenerate into a short circuit is so very long compared with the time involved in the oscillation of the fundamental frequency that it can be safely assumed that the transformer winding oscillates through a complete cycle of the fundamental before the voltage applied to the terminals of the transformer windings can be appreciably lowered.

Natural Frequency of Oscillation of a Winding

Formula (35e) is useful for the purpose of obtaining the natural frequency of a winding. For a solenoid consisting of a single one-layer coil, the capacitance between turns may be neglected and the formula becomes.

$$f = \frac{2\,\pi\,n^2}{\sqrt{L\,C_g}}. \tag{35f}$$

From this equation we reach the very interesting conclusion that the natural frequency of oscillations are not simple multiples of the fundamentals in the order of 1, 2, 3, 4, etc., but that they are proportional to the square of these values that is 1, 4, 9, 16, etc., (neglecting the variation of inductance with wave length.)

If the capacitance between turns or coils is not negligible the formula reverts to equation (35e). The influence of coil capacitance is to make the natural frequencies lower than those given by formula (35f). Moreover, the harmonies are not simple multiples of the fundamental in every case.

At very high frequencies the influence of ground capacitance becomes negligible compared with series capacitance and the formula for frequency becomes,

$$f = \frac{n}{\sqrt{L\,C_s}} \tag{35g}$$

Damping Factors

Of course the voltage distribution as given in Fig. 5 departs considerably from what occurs in practise on account of dielectric losses, which introduce a damping factor, which is greater the higher the frequency.

It may be, therefore, especially where the order of frequencies progresses as the square of the wave lengths, that before the fundamental has completed one-half of a complete oscillation most of the higher frequencies have become negligible. A knowledge of the dielectric losses would be necessary to determine whether the voltage stresses are appreciably restricted by them.

High-Frequency Transients

An idea of the transient phenomena occurring within a transformer winding when sustained high frequency is impressed can best be obtained by considering the applied high frequency to be made up of successive positive and negative rectangular waves, the frequency of which is the same as the fundamental frequency of the transformer winding. During the time of the first half cycle, phenomena are as described above. When the applied wave changes from positive to negative, the voltage to ground within the transformer winding due to the fundamental has reached its maximum value. The effect of this change in applied voltage is to increase the amplitude of the fundamental approximately by twice the amplitude of the applied voltage. In the example given the above fundamental has a value of 1.24 times the terminal applied voltage for the case of isolated neutral. At each cycle this value is increased by twice the value of the terminal voltage, causing the voltage to ground to increase regularly in steps so that the maximum potential to ground at the end of the second half cycle is 4.72, at the end of the third half cycle 7.20 etc.

The above assumes that the internal losses are insufficient to reduce its amplitude and neglects the amplitude of all other harmonics. Of course, if the applied frequency happens to be equal to the frequency of one of the harmonics the building up of voltage by resonance as described above would apply to that harmonic and not to the fundamental.

It should be noted that the amplitude of the oscillations within the transformer winding are not necessarily the amplitude of the impressed voltage, but they depend upon the amplitude of impressed wave voltage and also upon the values given in the table under "Transient Voltage Distribution." These values are dependent upon the value of "α"; the smaller the value of "α" the smaller will be the value of internal oscillations, and moreover, for a given value of "α" the amplitudes of the oscillations are smaller for the shorter wave lengths.

However, when a sustained high frequency, or a damped oscillation, having a frequency equal to a natural frequency of the transformer, is impressed, the resulting stress as previously shown may very greatly exceed those given by the curves, the values being limited by the value of "α", the internal losses, and the damping factor of the impressed wave. For a reliable quantitative analysis an accurate knowledge of all these factors is necessary.

IV.—Impulse Tests on Small High-Voltage Transformer Winding

A 30,000-volt transformer winding made up of a stack of 52 disk coils approximately 15 in. long and 10 in. in diameter was

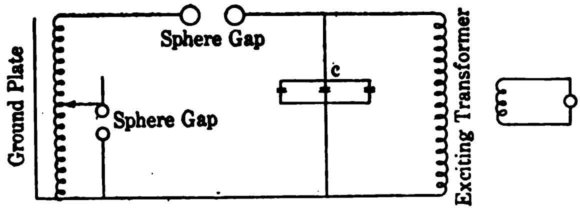


Fig. 8—Impulse Tests on Small High-Voltage Transformer-Connection Diagram

used for these tests. Grounded metal cylinders were placed inside and outside of the stack, to simulate low-voltage winding and tank.

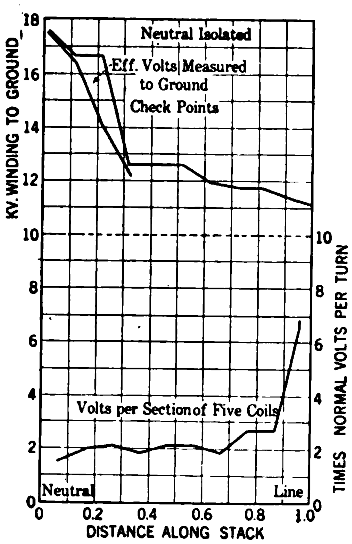


Fig. 9—Impulse Tests on Small High-Voltage Transformer—for Connections See Fig. 8

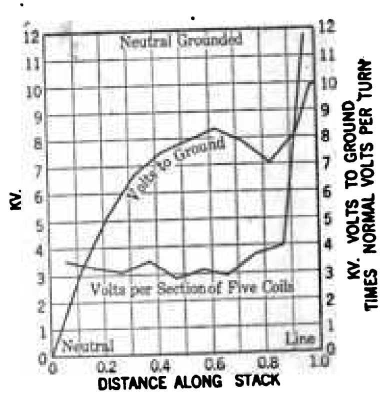


Fig. 10—Impulse Tests on Small High-Voltage Transformer

The connection used in testing is shown in Fig. 8. Voltage was applied from line to ground; in some tests the neutral (one end) of the coil was grounded to the metal cylinders and in the other test isolated. A spark gap A was connected in series with the coil, and a large condenser C in multiple with it. The operation is as follows: As the voltage rises, it charges the condenser C. This voltage also appears across the gap A, a very small part being consumed in the capacitance of the coil. When the gap arcs over, the entire voltage, abruptly applied across the winding, gives rise to a series of oscillations, within the winding. The use of the condenser C (which must be as large as practicable) is to maintain the applied voltage while the capacitance of the coil is being charged and also to allow an easy return path for the high-frequency oscillations set up

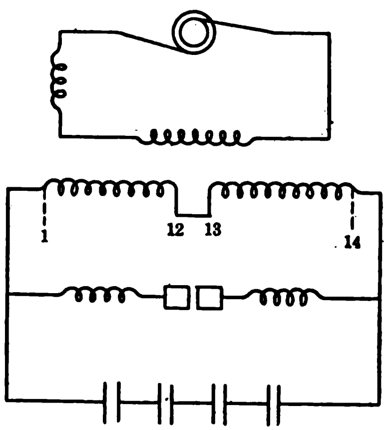


FIG. 11—HIGH-FREQUENCY INVESTIGATION—60 CYCLES—3000 KV-A.—140,000 VOLTS—DIAGRAM OF CONNECTIONS

in the coil. As these oscillations die out before the 60-cycle voltage has appreciably changed, the impressed voltage is in effect a sustained rectangular wave.

The voltage to ground and the voltage gradients are plotted in Fig. 9 and Fig. 10.

Constants of the Coil

Ground capacitance $C_g = 12 \times 10^{-10}$ farads.
Series capacitance $C_s = 4.4 \times 10^{-12}$ farads.

$$\alpha = \sqrt{\frac{C_g}{C_s}} = 16.5$$

V.—High-Frequency Tests*

High-frequency tests on the high-voltage windings of 3000-kv-a. 140,000-volt transformer (Fig. 4) compared with theoretically calculated stresses resulting from impact of rectangular waves.

Transformer constants;

C_g = Ground capacitance = 9×10^{-10} farads.
C_s = Coil capacitance = 9×10^{-12} farads.

$$\alpha = \sqrt{\frac{C_g}{C_s}} = 10$$

Connection diagrams for these tests are given in Fig. 11, and

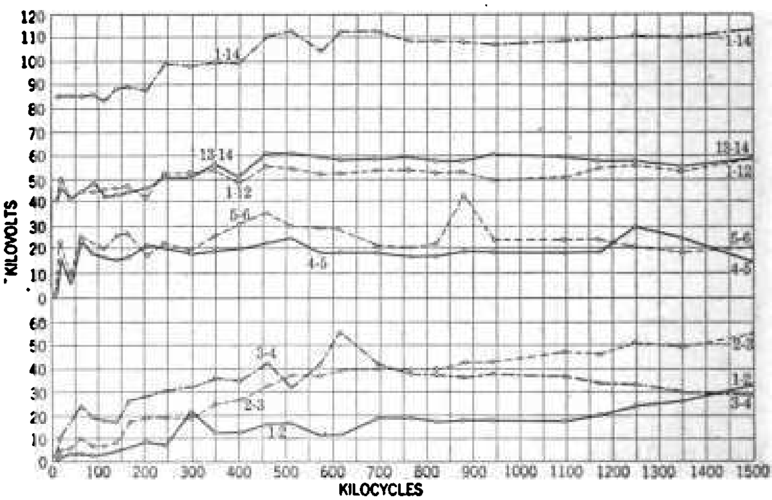


Fig. 12—High-Frequency Voltages Measured between Taps

high-frequency voltages obtained across taps are plotted in Fig. 12 and Fig. 13. In Fig. 14 is plotted the approximate volts obtained by these tests at any frequency (curve marked by dots). For comparison the volts per turn as theoretically obtained for rectangular waves (See Fig. 6) and corrected for end turn insulation has been plotted on the same sheet (curve marked by crosses.) The theoretical values have been multiplied by two, for the following reason.

Discharge takes place every one-half wave at the maximum

*Tests made by J. M. Weed and J. E. Clem.

point of the wave. Condensers discharge into the inductive circuit, the maximum value of high-frequency voltage being equal to maximum value of the 60-cycle voltage. One-half

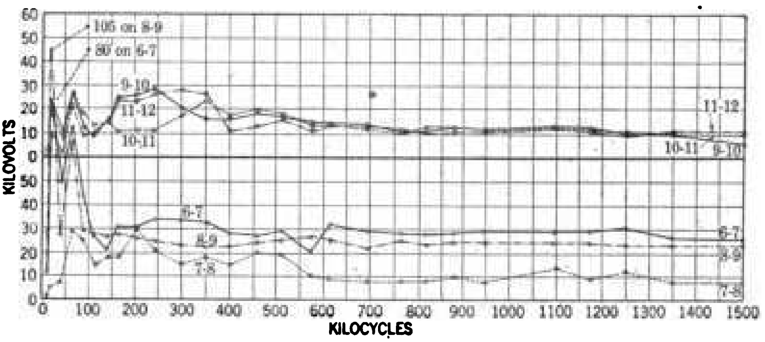


Fig. 13—High-Frequency Voltages Measured between Taps

wave of the high-frequency discharge is shown in Fig. 15. It is seen that during this half cycle the high-frequency voltage changes from maximum positive to maximum negative. The original 60-cycle voltage distribution, just before the arc-over,

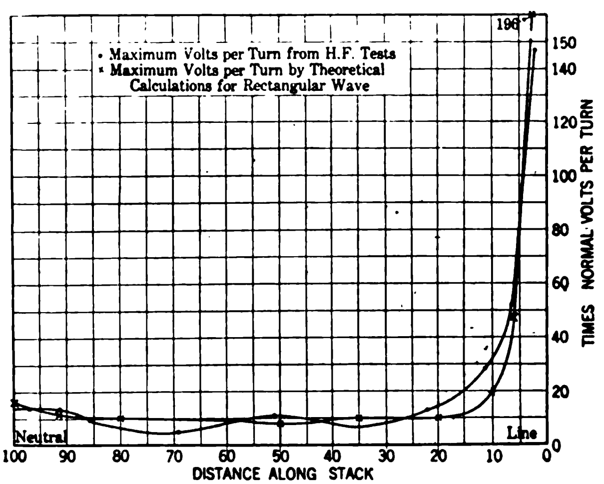


Fig. 14—3000-Kv-a.—140,000-Volt Transformer

is the straight line shown in Fig. 16. Half a high-frequency cycle after the arc-over, the resultant of the 60-cycle and high-frequency voltage distribution is shown by curve "*a*" in Fig. 16, assuming that the frequency is high enough for the winding to act as condenser for the first half cycle of the high-frequency wave. Thus it is evident that the combination of the high- and low-frequency waves doubles the voltage gradient.

HIGH-FREQUENCY TEST DATA

Taps	Per cent turns	Per cent along stack	Maximum volts per turn times normal
1—2	.19	2.1	147 at 1,500,000 Cycle
2—3	1.2	6.3	52 1,500,000 "
3—4	4.	11.4	28 at 600,000 "
4—5	8.2	16.7	11 at 62,000 "
5—6	13.5	22.5	13.5 at 900,000 "
6—7	30.7	36	5.6 62,000 "
7—8	48	51.5	11.5 62,000 "
8—9	68	69	6 20,000 "
9—10	89	86	9.9 62,000 "
10—11	94	91.5	13 62,000 "
11—12	98	97	14 62,000 "

The volts per turn occurring at ends of windings are the same as if a rectangular wave of voltage 2 E had been applied. For this reason the values for volts per turn as theoretically obtained for rectangular waves have been multiplied by two in order to

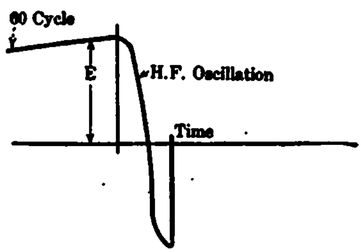


FIG. 15—TIME-VOLTAGE CURVE AT LINE END OF WINDING

compare these values with the high-frequency test. The higher the frequency of test, the more nearly should the condenser volts per turn distribution equal twice the voltage distribution caused by a rectangular wave.

V—Mathematical Development

1. Initial voltage distribution in a winding for a steep wave front.

2. Initial effective or equivalent capacitance of a winding for a steep wave front.

3. Time required to charge the initial effective capacitance of a winding.
4. Analysis of transient phenomena following the initial. voltage distribution.
5. Surge impedance of an oscillating winding.
6. Example.

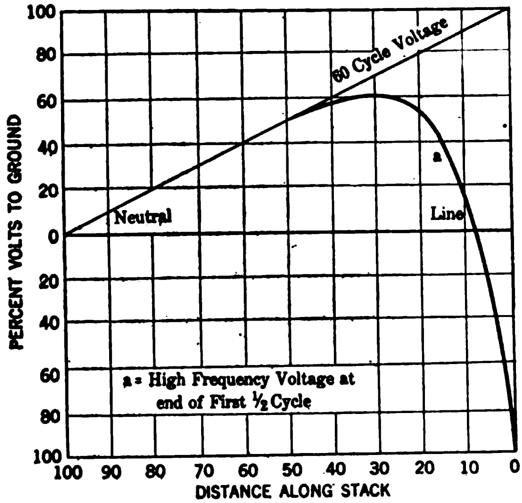


Fig. 16—Space-Voltage Curves for $t = 0$ and $t = \frac{1}{2}$ High-Frequency Cycle

1. Initial Voltage Distribution in a Winding for Steep Wave Fronts

If voltage is suddenly applied to a system of inductances in multiple with a system of capacitances, the initial voltage distribution will be determined by the capacitances. At the

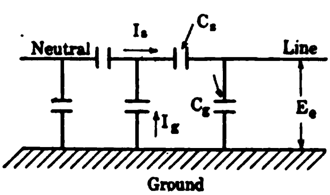


Fig. 17

instant of impact, the condensers short-circuit the steep wave front. If the wave is maintained, the condensers become charged, and the voltage rises. The time required for this charging is extremely small as given by equation (22).

The voltage distribution, when the condensers are charged, is the same as that when an alternating voltage is applied to the system of condensers with the inductance disconnected. Considering the system of internal and ground capacitances by themselves (Fig. 17), and assuming an alternating voltage of frequency f, $\left(f = \frac{\omega}{2\pi}\right)$, applied, the following relations will be evident.

Let C_s = Capacitance between line end of coil and neutral, shunting the winding.

C_g = Capacitance between winding and ground for a coil length of line to neutral.

l = Coil length line to neutral.

$l\,C_s$ = Capacitance between portions of winding unit distance apart.

$\frac{C_g}{l}$ = Capacitance to ground per unit length of coil

e = Voltage to ground at any point.

$\frac{d\,e}{d\,x}$ = Voltage gradient, volts per unit length of coil.

E_0 = Impressed voltage line to ground.

ω = $2\,\pi\,f$ = angular velocity.

I_s = Current in the coil capacitance C_s.

$$= \omega\,C_s\,l\,\frac{d\,e}{d\,x} \tag{1}$$

I_g = Current in ground capacitance C_g (total.)

$\frac{d\,I_g}{d\,x}$ = Current in ground capacitance per inch of coil.

$$= \omega\,\frac{C_g}{l}\,e \tag{2}$$

The increase per unit distance in the current of the shunt capacitance is the current supplied by the ground capacitance per unit length, that is,

$$\frac{d\,I_s}{d\,x} = \frac{d\,I_g}{d\,x} \tag{3}$$

Substituting in equation (3) the value of $\frac{d\,I_s}{d\,x}$ from (1), and

$\frac{d\,I_g}{d\,x}$ from (**2**), and simplifying,

$$\frac{d^2 e}{d\,x^2} - \frac{1}{l^2}\,\frac{C_g}{C_s}\,e = 0 \qquad \textbf{(4)}$$

The solution of this equation is of the form

$$e = A\,\epsilon^{px} \qquad \textbf{(5)}$$

Substituting (**5a**) and its derivative in (**4**), and solving for p,

$$p = \pm \frac{1}{l}\sqrt{\frac{C_g}{C_s}} \qquad \textbf{(6)}$$

and

$$e = A\,\epsilon^{px} + B\,\epsilon^{-px} \qquad \textbf{(7)}$$

The constants A and B are determined by the terminal conditions as follows.

Case I.—Neutral Isolated. If the neutral is isolated at

$x = 0 \quad I_s = 0$ (**8**)

The value of I_s is, from (**1**) and (**7**).

$$I_s = \omega\,l\,C_s\,\frac{d\,e}{d\,x} \qquad \textbf{(9a)}$$

$$= \omega\,l\,C_s\,(A\,\epsilon^{px} - B\epsilon^{-px}). \qquad \textbf{(9b)}$$

Substituting, $x = 0$, and $I_s = 0$

$$0 = \omega\,l\,C_s\,p\,(A - B) \qquad \textbf{(10a)}$$

$$A = B \qquad \textbf{(10b)}$$

Substituting, (**10b**) in (**7**),

$$e = A\,(\epsilon^{px} + \epsilon^{-px}) \qquad \textbf{(10c)}$$

$$= A'\cosh\,(p\,x) \qquad \textbf{(10d)}$$

Now, at $x = l \quad e = E_0$ (**10e**)

$$e = E_0\,\frac{\cosh\,(p\,x)}{\cosh\,(p\,l)} \qquad \textbf{(11a)}$$

$$= E_0\,\frac{\cosh\left(\frac{\alpha\,x}{l}\right)}{\cosh\,(\alpha)} \qquad \textbf{(11b)}$$

where $\alpha = p\,l = \sqrt{\frac{C_g}{C_s}}$ and the neutral is isolated

Case II.—Neutral Grounded. The terminal conditions are

at $x = 0 \qquad e = 0$ (**12a**)

and at $x = l \qquad e = E_0$ (**12b**)

Applying condition (**12a**) to equation (**7**).

$$0 = A + B$$

$$A = -B$$

$$e = A\,(\epsilon^{px} - \epsilon^{-px}). \quad \textbf{(12c)}$$

$$e = A' \sinh p\,x \quad \textbf{(12d)}$$

Applying condition (**12b**) to equation (**12d**)

$$E_0 = A' \sinh p\,l \quad \textbf{(12e)}$$

$$A' = \frac{E_0}{\sinh p\,l}$$

Therefore, $e = E_0 \dfrac{\sinh p\,x}{\sinh p\,l}$ **(13a)**

$$= E_0 \frac{\sinh\left(\dfrac{\alpha\,x}{l}\right)}{\sinh \alpha} \quad \textbf{(13b)}$$

where, $\alpha = p\,l = \sqrt{\dfrac{C_g}{C_s}}$ and the neutral is grounded.

Initial voltage distribution curves for both the isolated neutral and the grounded neutral are given in Figs. 2 and 3.

General Remarks on the Initial Voltage Distribution

When the capacitance to ground is negligible compared with the capacitance between coils, that is, $C_g = 0$, the initial voltage distribution is determined by the series capacitances and is a straight line if the capacitance between coils is uniform.

As the capacitance to ground more and more predominates, the curve becomes steeper near the line end, that is, the voltage is concentrated more and more near the line end.

The voltage gradient (volts per unit length) is $e_s = \dfrac{d\,e}{d\,x}$

$$= \frac{\alpha\,E_0 \sinh\left(\dfrac{\alpha\,x}{l}\right)}{l \cosh \alpha} \text{ neutral isolated} \quad \textbf{(14a)}$$

$$= \alpha\,\frac{E_0}{l}\,\frac{\cosh\left(\dfrac{\alpha\,x}{l}\right)}{\sinh \alpha} \text{ neutral grounded} \quad \textbf{(14b)}$$

The maximum voltage gradient is at the line end, $x = l$,

$$e_s = \frac{\alpha E_0 \sinh \alpha}{l \cosh \alpha} \text{ neutral isolated} \tag{15a}$$

$$= \alpha \frac{E_0}{l} \tanh \alpha \text{ neutral isolated} \tag{15b}$$

Similarly $e_s = \alpha \dfrac{E_0 \cosh \alpha}{l \sinh \alpha}$ neutral grounded (16a)

$$= \alpha \frac{E_0}{l} \coth \alpha \text{ neutral grounded} \tag{16b}$$

For α greater than 3,
Tanh α = coth α = 1
This is true for most transformers, and the voltage gradient at the line end becomes $\left(\dfrac{\alpha E_0}{l}\right)$ volts per unit length.

If the voltage were uniformally distributed throughout the winding, the gradient would have been $\left(\dfrac{E_0}{l}\right)$. Therefore, when the total capacitance to ground is very much larger than the coil capacitance between neutral and line, the voltage gradient at the line end is α times the value corresponding to uniform distribution.

It will also be noticed that for values of α greater than 5, the initial voltage distribution for a steep wave front is practically the same for the case of isolated as for grounded neutral.

Initial Effective Capacitance of a Winding for a Steep Wave Front.

In calculating the time of charging the system of coil and ground capacitances by an incoming wave, it becomes necessary to know the effective capacitance of the combination, This is equal to the effective capacitance for alternating voltage.

Representing this effective capacitance by C_{eff}, line current by I, line voltage (a-c.) by E, frequency by "f".

$$I = 2 \pi f C_{eff} E \tag{17a}$$

$$\text{or } C_{eff} = \frac{I}{2 \pi f E} \tag{17b}$$

$$\text{But } I = I_s \text{ at } x = l \tag{17c}$$

$$\text{and } I_s = 2 \pi f C_s l \left(\frac{d e}{d x}\right) \tag{17d}$$

The value of $\left(\frac{d\,e}{d\,x}\right)$ must be taken from equation of initial voltage distribution.

Case I.— Neutral Isolated. By equation (**15b**) at $x = l$

$$\frac{d\,e}{d\,x} = \alpha\,\frac{E}{l}\tanh\alpha \tag{18a}$$

$$\text{and } I = 2\,\pi\,f\,C_s \times \alpha\,\frac{E}{l}\tanh\alpha \tag{18b}$$

Substituting (**18b**) in (**17b**).

$$C_{eff.} = \sqrt{C_g\,C_s} \times \tanh\alpha\text{, neutral isolated} \tag{19}$$

Case II.—Neutral Grounded. Solving similarly for the above case, but using equation (**16b**) instead of (**15b**),

$$C_{eff} = \sqrt{C_g\,C_s}\,\coth\alpha \tag{20}$$

As stated before, for values of α greater than 3, which is the case with most transformers, $\tanh\alpha$ equals one, $\coth\alpha$ equals one, and

$$C_{eff} = \sqrt{C_g\,C_s} \tag{21}$$

whether the neutral is grounded or isolated.

Time Required to Charge the Condensers to the Initial Voltage Distribution

The time required to charge a condenser by an incoming wave is given by the following formula, modified from one given by G. Faccioli in *G. E. Review*, 1914, page 749.

$$t = 2.3\,C_{eff}\,Z\,\log_{10}\left(\frac{100}{100 - \%\,E_0}\right)\text{ seconds.} \tag{22}$$

$C_{eff.}$ = Equivalent massed capacitance of the winding from line to neutral, calculated by formula, (**19**), (**20**), or (**21**).

Z = Surge impedance of the transmission line, and may be taken between 300 and 400.

$\%\,E_0$ = The actual terminal voltage in per cent of the final terminal voltage.

As an example, we will calculate the time necessary to charge the capacitance of the transformer described above to 90 per cent of its final value.

$C_g = 9 \times 10^{-10}$ farads.
$C_s = 9 \times 10^{-12}$ farads
$C_{eff.} = 9 \times 10^{-11}$ by formula (**21**)

Substituting this value of C_{eff} in (22) we obtain

$$t = 2.3 \times 9 \times 10^{-11} \times 300 \times \log_{10}\left(\frac{100}{100-90}\right)$$

$= 6.2 \times 10^{-8}$ seconds, which is about one twenty millionth of a second. It is justifiable, then, to assume that the initial voltage distribution by the condensers takes place before the inductance can begin to draw any appreciable current.

Analysis of Transient Phenomena Following the Initial Voltage Distribution

Let I_L = Current in the winding.
I_s = Current in the coil capacitance.
I_g = Current in the ground capacitance (total).

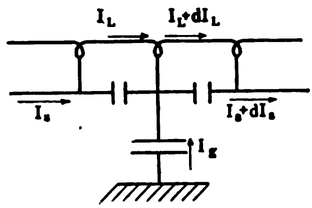


Fig. 18

The following relation will be evident from Fig. 18.

$$\frac{d\,I_L}{d\,x} + \frac{d\,I_s}{d\,x} - \frac{d\,I_g}{d\,x} = 0 \tag{23a}$$

Equation (**23a**) states that the increase in the currents in the winding and the internal capacitance (as we go towards the line end) is equal to the current contributed by the ground capacitance, which is evident from Fig. 18.

For reasons that will be seen later, we differentiate (**23a**) with respect to t, and obtain

$$\frac{d^2 I_L}{dxdt} + \frac{d^2 I_s}{dxdt} - \frac{d^2 I_g}{dxdt} = 0 \tag{23b}$$

It will be noticed that equations (**23a**) and (**23b**) are true generally and are independent of what the initial or final voltage distributions may be.

Equation (23) involves three variables, I_L, I_g and I_s, and, before it can be solved, it must be expressed in one variable.

Since we are interested mostly in the voltages, we will express the variables in terms of voltages.

Let e = voltage to ground.

e_s = voltage gradient (volts per inch of coil) in the winding.

$= \dfrac{d\,e}{d\,x}$

N = Total turns (line to neutral)

φ = Total flux linking any turn.

φ_L = Leakage flux linking any turn.

φ_M = Main flux linking the total coil.

B_L = Leakage flux density at any point.

h = Effective length of leakage path.

L = Leakage inductance of coil.

(MLT) = Mean length of turn.

(For a complete list of symbols see the end of this paper.)

To Find the Relation Between I_s and Voltage. The current I_s in the capacitance between coils is equal to the capacitances per inch $l\,C_s$ times the rate of change of voltage per inch $\left(\dfrac{d\,e_s}{dt}\right)$..

$$I_s = l\,C_s \frac{d\,e_s}{dt} = C_s\,l \;\frac{d^2\,e}{dxdt} \tag{24}$$

$$\frac{d^2\,I_s}{dxdt} = C_s\,l\;\frac{d^4\,e}{dx^2\,dt^2} \tag{25}$$

Equation (25) can be directly substituted in equation (23b).

To Find the Relation Between I_g and Voltage. The current density in the capacitance to ground is equal to capacitance to ground per inch of coil times the rate of change of voltage to ground.

$$\frac{d\,I_g}{d\,x} = \frac{C_g}{l}\;\frac{d\,e}{dt} \tag{26}$$

and

$$\frac{d^2\,I_g}{dxdt} = \frac{C_g}{l}\;\frac{d^2\,e}{dt^2} \tag{27}$$

Equation (27) can be directly substituted in (23b).

To Find the Relation Between I_L and Voltage. The relation between the magnetizing current (I_L) and the voltage is somewhat more complicated than that of the capacitance currents and voltage, due to the mutual inductance between turns and the presence of leakage and main fluxes. The connecting links between current and voltage to ground at any point x are current at x, ampere turns or magnetomotive force effective in producing leakage flux, flux density at point x, flux producing voltage at point x, volts per turn and volts. See Fig. 19. The mathematical relation between these physical quantities are as follows:

The voltage gradient e_s, *i. e.*, volts per inch, equals number of turns per inch N/l times the rate of change of total flux φ (= main flux φ_m + leakage flux φ_L). Thus,

$$e_s = \frac{N}{l} \frac{d\varphi}{dt} 10^{-8} \tag{28a}$$

$$= \frac{N}{l} 10^{-8} \frac{d}{dt} \left\{ \varphi_L + \varphi_m \right\} \tag{28b}$$

The leakage flux (φ_L) linking x is obtained by integrating the leakage flux density (B_L). Thus,

$$e_s = \frac{N}{l} 10^{-8} \frac{d}{dt} \left\{ (MLT) \int_x^l B_L \, dx + \varphi_m \right\} \tag{28c}$$

If we differentiate (28c) with respect to x we get rid of the integral sign, and also of the last term because it is independent of x. Thus,

$$\frac{d e_s}{d x} = - \frac{N}{l} 10^{-8} (MLT) \frac{d B_L}{dt} \tag{29a}$$

The negative sign in (29a) is due to x being the negative limit of integration in (28c). Of course it could have been assumed of the opposite sign provided other signs were kept consistent with it.

The leakage flux density B_L in (29a) is produced by the ampere turns enclosed by the flux.

$$B_L = \frac{N\mu}{lh} \int_0^x I_L \, dx \tag{29b}$$

where $\mu = 4 \frac{\pi}{10}$ when using dimensions in cm.

$= 3.2$, when using dimensions in inches.

h = effective length of leakage path.

It may be questioned why in (**28c**) we use the limits $\int_x^l$ and in (**29b**) $\int_0^x$. The reason may be stated thus; the flux linking a current is outside of it, and the ampere turns producing the flux are inside of it. Therefore, if the total flux linking a certain point be integrated on the right of that point, the ampere turns producing that flux must be integrated on the left of the point. As a concrete example consider Fig. 19.

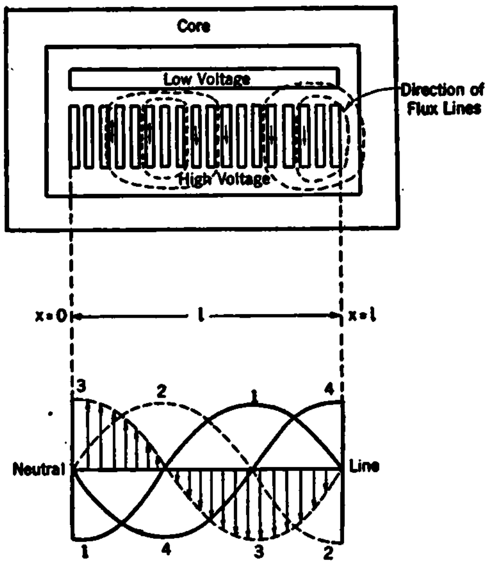


FIG. 19—NATURAL OSCILLATION WITHIN TRANSFORMER WINDING FOR ¾ WAVE LENGTH WITH ISOLATED NEUTRAL AND LINE END CONNECTED TO TRANSMISSION LINE

Curve 1, Volts to ground = E. Curve 2, Volts per turn = $D\,E/D\,X$. Curve 3, Flux density, or Magnetomotive force = $D^2\,E/D\,X^2$. Curve 4, Current = $D^3\,E/D\,X^3$.

The flux producing voltage in a turn at the point x is all the flux from x to l; but the ampere turns producing the flux at x are all the ampere turns from 0 to x.

Returning to the main problem, (**29b**) substituted in (**29a**) gives

$$\frac{d\,e_s}{d\,x} = -\frac{3.2\,N^2\,(MLT)}{l^2\,h\,10^8}\,\frac{d}{d\,t}\int_0^x I_L\,d\,x \qquad \textbf{(29c)}$$

$$= \frac{L}{l^3}\,\frac{d}{d\,t}\int_0^x I_L\,d\,x \qquad \textbf{(29d)}$$

where

$$L = \frac{3.2\,l\,N^2\,(MLT)}{h\,10^8} \text{ (See note opposite)} \qquad \textbf{(29e)}$$

Differentiating (**29d**) with respect to x eliminates the integral sign. Thus,

$$\frac{d^2 e_s}{d x^2} = - \frac{L}{l^3} \cdot \frac{d I_L}{dt} \qquad \textbf{(30a)}$$

Remembering that e_s (voltage gradient) is equal to $\frac{d e}{d x}$ (derivative of voltage to ground), (**30a**) becomes

$$\frac{d^3 e}{d x^3} = \frac{L}{l^3} \frac{d I_L}{dt} \qquad \textbf{(30b)}$$

(**30b**) gives the value of $\frac{d I_L}{dt}$, while, equation (**23a**) involves $\frac{d I_L}{d x}$. If we differentiate the former with respect to x and the latter with respect to t, substitution becomes possible. This is the reason why (**23a**) was differentiated with respect to time giving (**23b**).

Differentiating (**30b**) with respect to x and rearranging.

$$\frac{d^2 I_L}{dxdt} = \frac{-l^3}{L} \frac{d^4 e}{d x^4} \qquad \textbf{(31)}$$

(**31**), (**25**) and (**27**) express in terms of voltage the three terms of the fundamental differential equation (**23b**). Making the substitution, (**23b**) becomes

---

NOTE:—In discussing the leakage flux relations, we have assumed the effective length of the path as constant and independent of wave length. This is approximately true for only short waves on disk coils when the radial build of the coils is comparable to the wave length. Otherwise, the formula for L will not be accurate enough for numerical calculations. L, however, can be determined experimentally for various wave lengths as follows. Take one portion of the winding equal to one quarter of the wave length under consideration and short-circuit it. Take an equal portion adjacent to it, excite it as primary, and measure its inductance (= Reactance/377 for 60-cycle test). Multiply this value of inductance by 32 n^3 to get the effective value of L for that wave length.

If the winding consists of disk coils, for approximate calculations the effective length of leakage path may arbitrarily be assumed as equal to the radial coil-build plus one-sixth of wave length:

$$h = b + \frac{l}{6\,n} \qquad \textbf{(29f)}$$

$$\frac{d^4 e}{d x^4} - \frac{L C_s d^4 e}{l^2 dx^2 dt^2} + \frac{L C_g}{l^4} \frac{d^2 e}{dt^2} = 0 \qquad (32)$$

(32) is the general differential equation of electrical phenomena occurring in windings, except that it does not take into account resistance and dielectric losses.

A solution of this equation is

$$e = A \cos p\, x \cos \omega\, t. \qquad (33)$$

Substituting in equation (32) gives

$$\omega = \pm \frac{p^2 l^2}{\sqrt{L\,(C_g + l^2 p^2 C_s)}} \qquad (34)$$

These equations show that a transformer is capable of sinusoidal oscillations in space and time at various frequencies and wave lengths.

Determination of the Constants. The voltage distribution may be expressed as the sum of a transient and a permanent

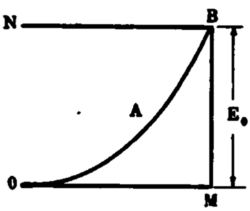


Fig. 20

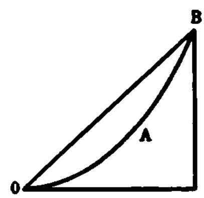


Fig. 21

component. The permanent component is the final voltage distribution. For a d-c. voltage applied, the final voltage distribution is $N B$ (Fig. 20) for the case of neutral isolated, that is, the whole coil assumes the impressed voltage E_0 against ground. The actual initial voltage being given by the hyperbolic curve $O A B$, the transient component is the difference between $N B$ and $O A B$. Expressing the same relation in another way $O A B$ represents the actual voltage when referred to $O M$ as the axis of reference, but represents the transient component when referred to $N B$ (the line of equilibrium) as the axis of reference. For neutral grounded (Fig. 21) $O B$ represents the permanent or equilibrium line, and $O A B$ (referred to $O B$ as axis) represents the transient component of the voltage distribution.

To try to substitute the hyperbolic formula of initial voltage distribution in equation (33) leads to an indeterminate form, but if analyzed into space harmonics (with respect to the line of final voltage distribution), substitution is readily effected, and the actual voltage to ground is the sum of the harmonics and the voltage of the reference line to ground.

An inspection of Fig. 5 shows that for neutral isolated the initial voltage distribution curve $O\,A\,B$ is a quarter wave length (and its odd harmonics) with respect to the equilibrium line $N\,B$ with maximum voltage at the neutral. An inspection of Fig. 6 shows that for neutral grounded the initial voltage distribution curve $O\,A\,B$ is a half wave length (and its even and odd harmonics). The numerical values of the harmonics are derived by harmonic analysis.

Representing wave length by λ, initially, for neutral isolated,

$$e' = E_\lambda \cos \frac{2\,\pi\,x}{\lambda} \tag{35}$$

and for neutral grounded

$$e' = E_\lambda \sin \frac{2\,\pi\,x}{\lambda} \tag{35a}$$

where $E\,\lambda$ is the maximum value of the wave.

The value of p in equation (34b) is therefore $\frac{2\,\pi\,x}{\lambda}$ and

$$\omega = \pm \frac{4\,\pi^2\,l^2}{\lambda^2 \sqrt{L\left(\frac{C_g + 4\,\pi^2\,l^2\,C_s}{\lambda^2}\right)}} \tag{35b}$$

$$= \pm \frac{4\,\pi^2\,(l/\lambda)^2}{\sqrt{L\,\{\,C_g + 4\,\pi^2\,C_s\,(l/\lambda)^2\,\}}} \tag{35c}$$

Now l/λ is the ratio of coil length to wave length and we shall designate it by small n. The longest wave to which the winding will oscillate is a quarter wave length, and $n = 1/4$. Thus n may be looked upon as the order of the space harmonic. If n equals one, the wave length equals the coil length; if n equals two, the wave length is one-half of the coil length, or, there are two complete wave lengths from line to neutral. Substituting n for l/λ,

$$\omega = \frac{4\,\pi^2\,n^2}{\sqrt{L\,(C_g + 4\,\pi^2\,n^2\,C_s)}} \tag{35d}$$

$$f = \frac{2\,\pi\,n^2}{\sqrt{L\,(C_g + 4\,\pi^2\,n^2\,C_s)}} \tag{35e}$$

For long wave lengths (low harmonics, n small) capacitance to ground predominates,

and
$$f = \frac{2\,\pi\,n^2}{\sqrt{L\,C_0}} \tag{35f}$$

For the very short wave lengths (high order of harmonics, n large) the coil capacitance C_s predominates.

$$f = \frac{n}{\sqrt{L\,C_s}} \tag{35g}$$

Since there are two values of ω, one positive and the other negative, and since $\cos(+\omega)$ equals $\cos(-\omega,)$ equation (33) becomes

$$e' = A\cos(2\,\pi\,n\,x/l)\cos(\omega_n t) + B\cos(2\,\pi\,n\,x/l)\cos(-\omega_n t) \tag{33a}$$

$$= A\cos(2\,\pi\,n\,x/l)\cos\omega_n t + B\cos(2\,\pi\,n\,x/l)\cos\omega_n t \tag{33b}$$

$$= (A + B)\cos(2\,\pi\,n\,x/l)\cos\omega_n t \tag{33c}$$

$$= E_n\cos(2\,\pi\,n\,x/l)\cos\omega_n t \tag{33d}$$

e' is the voltage to the axis of oscillation, and for every harmonic there is a term like (33d). The voltage to ground is, for neutral isolated,

$$e = E_0 - E_n\cos(2\,\pi\,n\,x/l)\cos\omega_n t \tag{33e}$$

and for neutral grounded

$$e = E_0\,(x/l) - E_n\sin(2\,\pi\,n\,x/l)\cos\omega_n t \tag{33f}$$

As the voltage oscillates above and below the line of equilibrium, and as the frequency is different for the different waves, we may assume that at some instant the maxima of all the harmonics at a point will add arithmetically, and that will represent the maximum voltage to ground at that point. That is, maximum voltage to ground equals, for neutral isolated.

$$\max e = E_0 + E_n\cos(2\,\pi\,n\,x/l) + \ldots \tag{36a}$$

and for neutral grounded

$$\max e = E_0\,(x/l) + E_n\sin(2\,\pi\,n\,x/l) + \ldots \tag{36b}$$

Of equal importance is the voltage gradient (volts per turn). The maximum voltage gradient at a point occurs when maximum voltage gradients of the various harmonics at that point add directly. This gives, for neutral isolated,

$$\max\frac{d\,e}{d\,x} = (2\,\pi\,n/l)\,E_n\sin(2\,\pi\,n\,x/l) + \ldots \tag{37a}$$

and for neutral grounded

$$\max \frac{d\,e}{d\,x} = E_0/l + (2\,\pi\,n/l)\,E_n \cos(2\,\pi\,n\,x/l) + \ldots \qquad \text{(37b)}$$

It is to be noted that although the higher harmonics may be small in magnitude yet the volts per turn which they produce may not be negligible on account of the volts per turn varying directly with the order of the harmonic. For instance, a tenth harmonic having a magnitude of ten per cent of the fundamental will cause as much strain per turn as the fundamental.

The initial voltage gradient was discussed under equations (14,) (15) and (16).

Surge Impedance of a Winding

The surge impedance of a transformer winding may be defined as the ratio of voltage impressed to current at line end of winding.

By equation (23)

$$\frac{d\,I_L}{d\,x} = \frac{d\,I_g}{d\,x} - \frac{d\,I_s}{d\,x}$$

$$= \frac{C_g\,d\,e}{l\,dt} - l\,C_s\,\frac{d^2\,e}{dxdt}$$

Substituting the general value of e,
that is, $e = E \cos p\,x \cos \omega\,t$

$$\frac{d\,I_L}{d\,x} = -\,\omega\,E\,(C_g/l + l\,C_s\,p^2) \cos p\,x \sin \omega\,t$$

and integrating

$$I_L = \frac{\omega\,E}{p}\,(C_g/l + l\,C_s\,p^2) \sin p\,x \sin \omega\,t \qquad \text{(38a)}$$

At the line end, $\sin p\,l$ equals one, and the maximum of the current occurs when $\sin w\,t$ equals one.

$$\max I_L = \frac{w\,E}{p}\,(C_g/l + l\,p^2\,C_s) \qquad \text{(38b)}$$

$$Z = \frac{E}{I_L} = \frac{p\,l}{\omega\,(C_g + l^2\,p^2\,C_s)} \qquad \text{(39a)}$$

$$Z = \frac{1}{2\,\pi\,n} \times \sqrt{\frac{L}{C_g + 4\,\pi^2\,n^2\,C_s}} \qquad \text{(39b)}$$

It is, therefore, evident that the surge impedance of a winding is not a constant but a function of the wave length, the shorter the wave length the less the surge impedance and hence, for a given voltage the current associated with higher harmonics is greater.

The current surges back and forth into the line, and it is assumed that the surge impedance of the line is low enough not to impede this free oscillation as will be seen in the following example.

Example

The constants of the high-voltage winding of the above mentioned transformer are as follows,

$$C_g = 9 \times 10^{-10} \text{ farads}$$
$$C_s = 9 \times 10^{-12} \text{ farads}$$
$$\alpha = 10$$

By formula (29e)

$$\begin{aligned} L &= 8 \text{ for } n = 1/4 \\ &= 14 \text{ `` } n = 1/2 \\ &= 20 \text{ `` } n = 3/4 \\ &= 23 \text{ `` } n = 1 \\ &= 27 \text{ `` } n = 5/4 \\ &= 30 \text{ `` } n = 3/2 \\ &= 32 \text{ `` } n = 7/4 \\ &= 34 \text{ `` } n = 4/2 \end{aligned}$$

The initial voltage distribution is, by equations (11b) and (13b).

$$\% \, e = \frac{e}{E_0} = \frac{\cosh(10\, x/l)}{\cosh(10)} \quad \text{neutral isolated}$$

$$= \frac{\sinh(10\, x/l)}{\sinh(10)} \quad \text{neutral grounded}$$

These are plotted in Figs. 2 and 3 together with curves for other values of α.

Analyzing the curve for $\alpha = 10$ into its space harmonics with respect to the corresponding axis of oscillation, we find the following values:

Neutral Isolated.—

Order of harmonic	Number of wave lengths n	Voltage $\frac{E_n}{E_0}$
Fundamental	1/4	− 125 %
3rd	3/4	+ 34 %
5th	5/4	− 15 %
7th	7/4	+ 7 %
9th	9/4	− 3.5%
11th	11/4	+ 1 %

The frequencies corresponding to these harmonics, calculated by equation (35e), are

Order of harmonics	n	λ	ω	f	Ratio
Fundamental	1/4	168 in.	28,000	4,600	1
3rd	3/4	56 "	150,000	24,000	5.2
5th	5/4	34 "	310,000	49,000	10.6
7th	7/4	24 "	480,000	76,000	16.5

Expressing the voltage to ground in per cent of line voltage (line to ground) the complete equation is

$$\% e = \frac{e}{E_0} = 1 - 1.25 \cos\left(\frac{\pi}{2} \times \frac{x}{l}\right) \cos\ 28{,}000\, t$$

$$+ 0.34 \cos\left(\frac{3\pi}{2} \times \frac{x}{l}\right) \cos 150{,}000\, t$$

$$- 0.15 \cos\left(\frac{5\pi}{2} \times \frac{x}{l}\right) \cos 310{,}000\, t$$

$$+ 0.07 \cos\left(\frac{7\pi}{2} \times \frac{x}{l}\right) \cos 480{,}000\, t \text{ etc.}$$

The maximum possible voltage to ground at any point at any time is equal to

$$\text{max. } \% e = 1 + 1.25 \cos\left(\frac{\pi}{2} \times \frac{x}{l}\right)$$

$$+ 0.34 \cos\left(\frac{3\pi}{2} \times \frac{x}{l}\right)$$

$$+ 0.15 \cos\left(\frac{5\pi}{2} \times \frac{x}{l}\right)$$

$$+ 0.07 \cos\left(\frac{7\pi}{2} \times \frac{x}{l}\right) \text{etc.}$$

This is plotted in Fig. 6.

The maximum voltage gradient is the derivative of the above equation.

$$\text{max. } \%\, e_s = \frac{d\,e}{d\,x} = \frac{\pi}{2\,l} \times 1.25 \sin\left(\frac{\pi x}{2\,l}\right)$$

$$+ \frac{3\pi}{2\,l} \times 0.34 \sin\left(\frac{3\pi x}{2\,l}\right)$$

$$+ \frac{5\pi}{2\,l} \times 0.15 \sin\left(\frac{5\pi x}{2\,l}\right)$$

$$+ \frac{7\pi}{2\,l} \times 0.07 \sin\left(\frac{7\pi x}{2\,l}\right) \quad \text{etc.}$$

This is also plotted in Fig. 6. Calculating the voltage gradient at the line end by this formula we find that it is less than that obtained by equation (15b). This means that a sufficient number of harmonics are not included in the above formula. Neglecting the highest harmonics is permissible for the interior of the coil as they may be damped out before the highest possible maximum would be reached, but for the line end it is better to calculate the voltage gradient by the hyperbolic formula. This has been done in the calculation of the above curve.

Neutral Grounded. The initial voltage distribution for this case is practically the same as for isolated neutral (see Fig. 3, $\alpha = 10$). Analyzing this curve into space harmonics with respect to the slant line $\alpha = 0$, we find the following values:

No. of wave lengths on the stack n	$\frac{E_x}{E_0}$	ω	f	Ratio
½ −	.58	84,000	13,000	1
2 × ½ +	.23	230,000	37,000	2.8
3 × ½ −	.13	390,000	62,000	4.8
4 × ½ −	.06	580,000	92,000	7.1

The maximum voltage to ground is

$$\text{max. } \% \, e = \frac{x}{l} + 0.58 \sin\left(\frac{\pi x}{l}\right) + 0.23 \sin\left(\frac{2\pi x}{l}\right) + 0.13 \sin\left(\frac{3\pi x}{l}\right) + 0.06 \sin\left(\frac{4\pi x}{l}\right) \text{ etc.}$$

This is plotted in Fig. 7.

The maximum voltage gradient is the derivative of the above.

$$\text{max. } \% \, e_s = \frac{de}{dx} = \frac{1}{l} + \frac{0.58\pi}{l} \cos\left(\frac{\pi x}{l}\right) + 0.23 \times \frac{2\pi}{l} \cos\left(\frac{2\pi x}{l}\right) + 0.13 \times \frac{3\pi}{l} \cos\left(\frac{3\pi x}{l}\right) + 0.06 \times \frac{4\pi}{l} \cos\left(\frac{4\pi x}{l}\right) \text{ etc.}$$

This also is plotted in Fig. 7.

Surge Impedance. By equation (**39b**)

n = 1/4	1/2	3/4	1	1 1/4	1 1/2	1 3/4	2
Z = 60,000	38,000	29,000	22,000	17,000	14,000	11,500	9,600

Symbols

α = $\sqrt{C_g/C_s}$

A = Constant of integration

B = Flux density, also constant of integration.

C = Capacitance.

C_g = Capacitance to ground (total.)

C_s = Coil capacitance (total), from line end to neutral.

E_0 = Terminal voltage, *i. e.*, line to ground.

E_n = Max. value of the "n" th harmonic to the axis of oscillation.

e = Voltage to ground at any point at any instant.

e_s = Voltage gradient in the winding, volts per unit length, proportional to volts per turn.

f = Frequency cycles per second, $\omega/2\pi$.

g = Subscript meaning ground.
I = Current.
l = Length of coil (or coil stack), line to neutral.
N = Number of turns total, line to neutral.
n = Number of wave lengths on the winding for a given harmonic.
ω = Angular velocity, radians per second.
$= 2 \pi f$
Z = Surge impedance.