2.11.9 Show that the ideal $(3, x^3-x^2+2x-1)$ in $\mathbb{Z}[x]$ not a principal ideal

Let $I = (3, x^3-x^2+2x-1)$ and
let $I' = \{3f(x) + (x^3-x^2+2x-1)g(x) \mid f(x), g(x) \in \mathbb{Z}[x]\}$.

We claim $I = I'$:

For any $f(x), g(x) \in \mathbb{Z}[x]$ $3f(x), (x^3-x^2+2x-1)g(x) \in I$ (Since I is an Ideal).
Since I is closed under addition, $3f(x) + (x^3-x^2+2x-1)g(x) \in I$.
Thus, $I' \subset I$.

Furthermore, I' is an ideal.
Thus, if $h(x) \in I$, then $h(x) \in I'$ (because I is the intersection of all ideals containing 3 and x^3-x^2+2x-1 and I' is such an ideal.)
$\Rightarrow I \subset I'$
$\Rightarrow I = I'$.

Now, suppose I is a principal ideal. Then $I = g(x)\mathbb{Z}[x]$ for some $g(x) \in \mathbb{Z}[x]$.

$3 \in I \Rightarrow 3 = g(x)h(x)$ for some $h(x) \in \mathbb{Z}[x]$.

$\Rightarrow 0 = \deg 3 = \deg g(x) + \deg h(x) \Rightarrow \deg g(x) = 0$

$\Rightarrow g(x) = a$ for some $a \in \mathbb{Z}$ $a \neq 0$. $\Rightarrow I = a\mathbb{Z}[x]$.

Thus, for some $h(x) = a_nx^n + \dots + a_3x^3 + \dots + a_1x + a_0 \in \mathbb{Z}[x]$,

$x^3 - x^2 + 2x - 1 = ah(x) = aa_nx^n + \dots + aa_3x^3 + \dots + aa_1x + aa_0$

$\Rightarrow aa_3 = 1 \Rightarrow a = \pm 1$.

We can assume $a = 1$ (because $1\cdot\mathbb{Z}[x] = -1\cdot\mathbb{Z}[x]$).
We thus have $I = \mathbb{Z}[x]$.

Now, we claim $1 \notin I$ (thus contradicting $I = \mathbb{Z}[x]$):

Suppose $1 \in I$. Then $1 = 3f(x) + (x^3-x^2+2x-1)g(x)$ for some $f(x), g(x) \in \mathbb{Z}[x]$.
Let a_i, b_i denote the coefficients of x^i in $f(x)$ and $g(x)$ respectively.
Since $1 = 3f(x) + x^3g(x) - x^2g(x) + 2xg(x) - g(x)$ we have

$3a_0 - b_0 = 1 \Rightarrow b_0 \equiv 2 \pmod 3$.

Also, (denoting $\deg g(x)$ by n)

$$\left.\begin{array}{l} 3a_{n+3} + b_n = 0 \\ 3a_{n+2} - b_n + b_{n-1} = 0 \\ 3a_{n+1} + 2b_n - b_{n-1} + b_{n-2} = 0 \end{array}\right\} \Rightarrow \begin{array}{l} b_n \equiv 0 \pmod 3 \\ b_{n-1} \equiv 0 \pmod 3 \\ b_{n-2} \equiv 0 \pmod 3 \end{array}$$

and,

$3a_i - b_i + 2b_{i-1} - b_{i-2} + b_{i-3} = 0 \quad \forall\, 3 \leq i \leq n$.

(Via induction) $\Rightarrow b_0 \equiv 0 \pmod 3 \not\equiv 2 \pmod 3$. contradiction. $\Rightarrow 1 \notin I$ $\Rightarrow$ But $1 \in \mathbb{Z}[x]$. $I \neq \mathbb{Z}[x]$.

Contradiction. Thus, I is not a principal ideal. ∎