Calculus Proof: Chain Rule

Given that $f(x)$ and $g(x)$ are differentiable, and $h(x) = f(g(x))$,

$$h^{'}(x) = f^{'}\big(g(x)\big)g^{'}(x)$$

Proof:

Let g_1 be some x value and consider the following (thinking of $g_1$as an output of g(x)):

$$f^{'}(g_1) = \lim_{h \to 0} \frac{f(g_1 + h) - f(g_1)}{h}$$

Set $h = g_2 - g_1$ (difference between two outputs of g(x)):

$$f^{'}(g_1) = \lim_{g_2 \to g_1} \frac{f(g_2) - f(g_1)}{g_2 - g_1}$$

Now, one may be tempted to set $g_2 = g(x+h)\ and\ g_1 = g(x)$ as well as making the limit with h approaching zero. However, this cannot be done as we do not if

$$g(x+h) - g(x) \neq 0 \quad if \quad h \neq 0$$

To overcome the possibility of having a zero in the denominator, let $E(g(x+h))$ be the "error term" that represents the difference in the slope of $f^{'}(g(x))$ and the secant line going through $f(g(x+h))$ and $f(g(x))$. This piecewise function deals with the possibility of the denominator becoming zero.

$$E(g(x+h)) = \begin{cases} \dfrac{f(g(x+h) - f(g(x))}{g(x+h) - g(x)} - f^{'}\big(g(x)\big), & if\ g(x+h) \neq g(x) \\ 0, & if\ g(x+h) = g(x) \end{cases}$$

We see now that the difference between $f^{'}(g(x))$ at $g(x+h)$ and $f(g(x+h))$ is the following:

$$E\big(g(x+h)\big)(g(x+h) - g(x))$$

We also see that:

$$f\big(g(x+h)\big) - f\big(g(x)\big) = E\big(g(x+h)\big)\big(g(x+h) - g(x)\big) + f^{'}(g(x))(g(x+h) - g(x))$$

$$= \Big(E\big(g(x+h)\big) + f^{'}(g(x)\Big)(g(x+h) - g(x))$$

Hence,

$$\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} = E(g(x+h)) + f^{'}(g(x))$$

Considering:

$$\lim_{h\to 0} E(g(x+h)) = \begin{cases} \lim_{h\to 0}\left[\frac{f(g(x+h)-f(g(x))}{g(x+h)-g(x)} - f^{'}\left(g(x)\right)\right], if\ g(x+h) \neq g(x) \\ \lim_{h\to 0} 0\,, \qquad if\ g(x+h) = g(x) \end{cases}$$

$$\lim_{h\to 0} E(g(x+h)) = \begin{cases} f^{'}(x) - f^{'}(x), & if\ g(x+h) \neq g(x) \\ 0, & if\ g(x+h) = g(x) \end{cases}$$

$$\lim_{h\to 0} E(g(x+h)) = 0$$

Taking into account $f\big(g(x+h)\big) - f\big(g(x)\big) = \Big(E\big(g(x+h)\big) + f^{'}(g(x)\Big)\,(g(x+h)-g(x))$ we divide both sides by h.

$$\frac{f\big(g(x+h)\big)-f(g(x))}{h} = \Big(E(g(x+h)) + f^{'}(g(x))\Big)\frac{g(x+h)-g(x)}{h}$$

We take the limit as h approaches zero on both sides.

$$\lim_{h\to 0}\left(\frac{f\big(g(x+h)\big)-f(g(x))}{h}\right) = \lim_{h\to 0}\big(E(g(x+h)) + f^{'}(g(x))\big)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$\lim_{h\to 0}\left(\frac{f\big(g(x+h)\big)-f(g(x))}{h}\right) = \Big(\lim_{h\to 0} E(g(x+h)) + \lim_{h\to 0} f^{'}(g(x))\Big)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$\lim_{h\to 0}\left(\frac{f\big(g(x+h)\big)-f(g(x))}{h}\right) = \Big(0 + \lim_{h\to 0} f^{'}(g(x))\Big)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$\lim_{h\to 0}\left(\frac{f\big(g(x+h)\big)-f(g(x))}{h}\right) = \Big(\lim_{h\to 0} f^{'}(g(x))\Big)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$\frac{d}{dx}f\big(g(x)\big) = f^{'}\big(g(x)\big)g^{'}(x)$$