2 Symmetries and group theory

"You boil it in sawdust: You salt it in glue
You condense it with locusts and tape
Still keeping one principle object in view -
To preserve its symmetrical shape."

Fit the fifth, The Hunting of the Snark, L. Carroll

2.1 Groups

For a more complete discussion of group theory, especially towards its rôle in relativity, the reader is recommended F. Gürsey. *Introduction to group theory*, Relativity, Groups and Topology: Les Houches lectures, ed.: C. DeWitt & B. DeWitt.

We collect together some concepts to do with symmetries and their mathematical cousins groups.

Definition. *A set of objects* $\mathcal{G} = \{g_i\}_{i\in\iota}$ *equipped with a product operation* $\cdot : \mathcal{G} \times \mathcal{G} \mapsto \mathcal{G}$ *is called a* (abstract) group *if the following axioms are satisfied:*
i) closure, the product of any two elements in $\mathcal{G}$ *is again an element of* $\mathcal{G}$*;*
ii) associativity, if $a, b, c \in \mathcal{G}$ *then* $a \cdot (b \cdot c) = (a \cdot b) \cdot c$*;*
iii) identity, there exists an element $e \in \mathcal{G}$ *such that* $e \cdot g = g \cdot e = e$ *for all* $g \in \mathcal{G}$*;*
iv) inverse, to every $g \in \mathcal{G}$ *there exists a* $g^{-1} \in \mathcal{G}$ *such that* $g \cdot g^{-1} = e = g^{-1} \cdot g$*.*
The group is said to be Abelian *if* $[g, h] = 0$ *for all* $g, h \in \mathcal{G}$*; otherwise, the group is said to be* non-Abelian*. A* subgroup H *of* $\mathcal{G}$ *is a subset of* $\mathcal{G}$ *that is also a group.*

Consider the following simply example:

Example 2.1 (Characteristic table)**.** *Consider the set* $\{e, a, b\}$ *equipped with the following multiplication law:*

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

It is evident that this describes a finite Abelian group; moreover, note that as $a^2 = b$ and $a^3 = e$, the set $\{a\}$ describes the whole group.

Definition. *Let $\mathcal{G}$ be a group. A group element is called* independent *if it cannot be expressed as a product of other group elements. If a set $X \subset \mathcal{G}$ exists so that we may express any element $g \in \mathcal{G}$ as a product of elements in X we call X the the* generators *of the group $\mathcal{G}$.*

Example 2.2 $((\mathbb{Z}_n, +_n))$**.** *Let $n \in \mathbb{N}$ then the set $\mathbb{Z}_n = \{0, 1, \ldots, n-2, n-1\}$ equipped with modulo n addition $+_n$ is a group.*

Definition. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two groups whose product operations are denoted $\cdot$ and $*$ respectively; a mapping $h : \mathcal{G} \mapsto \mathcal{G}'$ is said to be a* (group) homomorphism *if $h(a \cdot b) = h(a) * h(b)$ for all $a, b \in \mathcal{G}$. A* (group) isomorphism *ι is a bijective map $\iota : \mathcal{G} \mapsto \mathcal{G}'$ such that ι and ι^{-1} are homomorphisms. Two groups are said to be* isomorphic *if there exists an isomorphism between them.*

Example 2.3. *The group featuring in example 2.1 is isomorphic to $\mathbb{Z}_3$ where the isomorphism is $e \mapsto 0$, $a \mapsto 1$ and $b \mapsto 2$.*

Example 2.4 (The circle $U(1)$)**.** *The set of all elements of the form $e^{i\theta}$ where $\theta \in \mathbb{R}$ equipped with the operator $e^{i\theta}e^{i\theta'} = e^{i(\theta+\theta')}$ where the addition is $\mod 2\pi$ form a group known as $U(1)$, the* one dimensional unitary group*. Moreover, the group is Abelian.*

The reader interested in how this coordinate method works for the surface of the sphere S^2 is directed to the description of *stereographic coordinates* in §5.1.1 of Nakahara M., *Geometry, topology and physics*, Institute of Physics publishing.

Our two examples are very different in nature. In particular, $U(1)$ is easily geometrically identified with the unit circle; however, there exists no such obvious interpretation of our first group. Moreover, the second group is continuous, whereas the first is the discrete. In fact the unit circle is an example of a *manifold* which we present the definition of:

Definition. *A $n-$*dimensional differentiable manifold *$\mathcal{M}$ is a topological space equipped with a family of pairs $\{\mathcal{U}_i, \varphi_i\}_{i\in\iota}$ where $\bigcup_i \mathcal{U}_i = \mathcal{M}$ and $\varphi_i : \mathcal{U}_i \mapsto \mathcal{O}_i$ are smooth and invertible maps from $\mathcal{U}_i$ into an open set $\mathcal{O}_i \subset \mathbb{R}^n$. The pair $(\mathcal{U}_i, \varphi_i)$ is called a* chart *while the whole family of such pairs is called an* atlas.

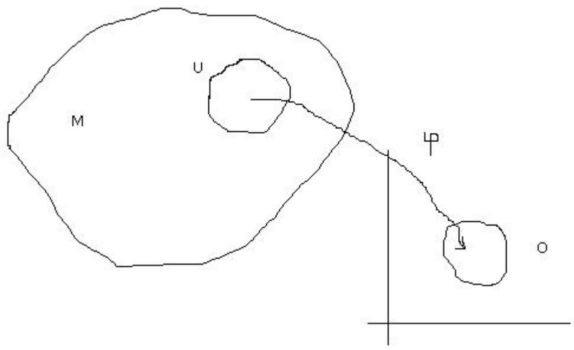

Figure 1: A manifold

Example 2.5 ($\mathbb{R}^n$)**.** *Euclidean space $\mathbb{R}^n$ is a manifold. The coordinate maps φ_i may be taken to be $\mathbb{1}$.*

Example 2.6 (The unit circle)**.** *Consider the unit circle S^1; for concreteness we take the circle $x^2 + y^2 = 1$ in the $x - y$ plane. Define $\varphi_1^{-1} : (0, 2\pi) \mapsto S^1$ and $\varphi_2^{-1} : (-\pi, \pi) \mapsto S^1$ by*

$$\varphi_i^{-1} : (\theta) = (\cos\theta, \sin\theta)^{\mathrm{t}} \quad \textit{and} \quad \varphi_2^{-1}(\theta) = (\cos\theta, \sin\theta)^{\mathrm{t}} .$$

Note that $\mathrm{Im}\varphi_i^{-1} = S^1 \setminus \{(1,0)\}$ *and* $\mathrm{Im}\varphi_2^{-1} = S^1 \setminus \{(-1,0)\}$.

The phrase 'topological space' need not concern us as it can be understood to mean 'space within which we have a notion of open set'. The reader uneasy with this definition need not worry as such fine geometric points do not pay a part in this course and are only mentioned for completeness as they enable us to make the following

Definition. *A* Lie group $\mathcal{G}$ *is a differentiable manifold which is endowed with a group structure.*

Again, we emphasise that all the Lie groups we shall be considering will have a matrix interpretation. Lie groups are of great importance in contemporary theoretical physics as they provide insight into the families of elementary particles.

Example 2.7 (Translations)**.** *The set* $\mathbb{R}$ *equipped with addition is an Abelian group, called the* (one dimension) translation group*, and it acts on position kets* $|x\rangle \in \mathbb{R}$ *by* $x\rangle \mapsto |g + x\rangle$*. More interesting is when the translation group acts on functions* f *by* $f(x) \mapsto f(g + x)$*. It is a consequence of Taylor's theorem that* $\exp(ig\hbar^{-1}p_x)f(x) = f(g + x)$ *where* p_x *is the momentum operator* $-i\hbar\partial_x$*. In this sense we say that* linear momentum is the (infinitesimal) generator of spatial translation*.*

It may happen that a Lie groups may be realised as the union of disjoint subsets; e.g. $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2$ such that $\mathcal{G}_1 \cap \mathcal{G}_2 = \emptyset$; e.g., a matrix Lie group may be partitioned by the sign of the determinant.

Definition. *A topological space is said to be* disconnected *if it is the union of two disjoint non-empty open sets; otherwise it is said to be* connected*. The* identity connected component $\mathcal{G}_0$ *of a Lie group* $\mathcal{G}$ *is a connected component containing the group identity element* e*.*

2.2 Representations and the general linear group

We now return to a discussion of our two examples. Our second example, being more abstract, naturally leads to the question how do we represent such elements? A convenient way is to consider a mapping of $\mathcal{G}$ onto a vector space V in such a way that one preserves the group product structure.

Definition. *A representation of a group* $\mathcal{G}$ *is the pair* $\{\iota, V\}$ *where* V *is a vector space which is also a group and* $\iota : \mathcal{G} \mapsto V$ *is a homomorphism.*

As the mapping ι preserves the group structure it is commonplace to treat $\mathcal{G}$ and its representations as more or less the same thing; however, one should be wary that while the group theory usually will be telling us something deep about a physical system the choice of representation is entirely imposed by us and we must be careful that such choices to do affect the physics.

In finite $n-$dimensional problems it is typical to take V to be the set of all $n \times n$ invertible matrices.

Example 2.8 (The general and special linear groups)**.** *Let* $\mathcal{F}$ *be a field (typically* $\mathbb{R}$ *or* $\mathbb{C}$*) then set of all* $n \times n$ *invertible matrices over* $\mathcal{F}$ *is denoted* $\mathrm{GL}_n(\mathcal{F})$ *which, when* *equipped with matrix multiplication, is easily shown to be a group known as the* general linear group. *The subgroup of* $\mathrm{GL}_n(\mathcal{F})$ *defined by* $\{M \in \mathrm{GL}_n(\mathcal{F}) \mid \det M = +1\}$ *is known as the* special linear group $\mathrm{SL}_n(\mathcal{F})$.

That $\mathrm{GL}_n(\mathcal{F})$ is a group is easily verified: clearly, $\mathbb{1} \in \mathrm{GL}_n(\mathcal{F})$, matrix multiplication is associative and any element of $\mathrm{GL}_n(\mathcal{F})$ has a unique inverse; if $M, N \in \mathrm{GL}_n(\mathcal{F})$ their product MN is also invertible, with inverse $N^{-1}M^{-1}$, so the set is closed under matrix multiplication. Moreover, since $\det AB = \det A \det B$ we conclude that $\mathrm{SL}_n(\mathcal{F})$ is a group also.

Example 2.9. *The circle group* $U(1)$ *is defined by the relation that* $z(\theta)z(\phi) = z(\theta+\phi)$ *where* $z(\theta) = e^{i\theta}$*; a* $\mathrm{SL}_2(\mathbb{R})$ *representation* M *of this group may be achieved by mapping*

$$M : U(1) \mapsto \mathrm{SL}_2(\mathbb{R}) \quad \textit{defined by} \quad M(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

These matrices satisfy $M(\theta)M(\phi) = M(\theta+\phi)$ *where the group action is matrix multiplication.*

In physics it is not the groups $\mathcal{G}$ themselves which are of interest but rather how they act on other objects. For example, $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathbb{R}^n$ by matrix multiplication. In general

Definition. *Let* $\mathcal{M}$ *be a set and* $\mathcal{G}$ *a Lie group. The* (left) group action L_g *of* $\mathcal{G}$ *on* $\mathcal{M}$ *is a map* $L_g : \mathcal{G} \times \mathcal{M} \mapsto \mathcal{M}$ *defined by* $L_g p = g \cdot p$ *where* $p \in \mathcal{M}$ *which satisfies: i)* $L_g L_h = L_{gh}$*; ii)* L_e *is the identity mapping where* e *is the group unit.*

Example 2.10 (The circle group $U(1)$)**.** *The left action of* $U(1)$ *on* $\mathbb{R}^2$ *may be realised by our matrix representation;* $\mathbf{x} \mapsto \mathbf{x}' = M(\theta)\mathbf{x}$.

Proposition 2.11. *For every fixed* $g \in \mathcal{G}$ *he left group action* L_g *is a bijection.*

Proof. For fixed g that $L_g : \mathcal{M} \mapsto \mathcal{M}$ is onto is easily seen, it remains to assert that every element in the image corresponds to a unique element in the range; again this is straightforward, one simply applies $L_{g^{-1}}$ to recover the element. $\square$

2.3 The orthogonal group

Groups are intimately related with symmetry properties. For example, we know that if we rotate a piece of paper that any straight line drawn on it has the same length irrespective of the orientation. To be more concrete let us consider $\mathbb{R}^2$, coordinatised by (x_1, x_2), and recall the Pythagorian invariant $x_1^2 + x_2^2$. We wish to consider the transformations $(x_1, x_2) \mapsto (x_1', x_2')$ such that $x_1^2 + x_2^2 = x_1'^2 + x_2'^2$; this relation may be equally written $x^i x^i = x'^i x'^i$ where summation of repeated indices is implied. Using matrix notation let

$$\mathbf{x} = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \quad \text{then} \quad \mathbf{x} \mapsto \mathbf{x}' = \mathcal{O}\mathbf{x}$$

which may be equally expressed in coordinate language as $x^i \mapsto x'^i = \mathcal{O}^{ij} x^j$; note that the position of the indices doesn't matter here as the metric is Euclidean (i.e. positive definite). In order to preserve the Pythagorian invariance we require that

$$x^j x^j = x'^j x'^j = \mathcal{O}^{ij} x^j \mathcal{O}^{ik} x^k \quad \Rightarrow \quad \mathcal{O}^{ij}\mathcal{O}^{ik} = \delta^{jk}$$

which may be written symbolically as

$$\mathcal{O}^{\mathrm{t}}\mathcal{O} = \mathbb{1}\,, \tag{1}$$

i.e., the inverse of $\mathcal{O}$ is simply given by its transpose; such matrices are called *orthogonal* and the totality of such 2×2 matrices we label $O(2)$.

Lemma 2.12. *Under matrix multiplication $O(2)$ is a group, known as the* orthogonal group *of 2×2 matrices.*

Proof. We proceed systematically, first by observing that the matrix product of any two 2×2 matrices is again a 2×2 matrix. Then, let $\mathcal{O}, \widetilde{\mathcal{O}} \in O(2)$ then $\mathcal{O}\widetilde{\mathcal{O}}$ satisfies (1). Indeed, $(\mathcal{O}\widetilde{\mathcal{O}})^{\mathrm{t}}\mathcal{O}\widetilde{\mathcal{O}} = \widetilde{\mathcal{O}}^{\mathrm{t}}\mathcal{O}^{\mathrm{t}}\mathcal{O}\widetilde{\mathcal{O}}$ from which satisfaction of the closure axiom follows. Associativity follows from the basic properties of matrix multiplication and it is trivial to see that $\mathbb{1} \in O(2)$. Finally, it remains to show that axiom (iv) is met. Relation (1) implies that, for each $\mathcal{O} \in O(2)$, $\det \mathcal{O} = \pm 1$; hence, every element of $O(2)$ has an inverse $\mathcal{O}^{-1}$ which we must show satisfies (1). The proof is completed by noting that $(\mathcal{O}^{-1})^{\mathrm{t}}\mathcal{O}^{-1} = \big(\mathcal{O}\mathcal{O}^{\mathrm{t}}\big)^{-1}$. Hence, $O(2)$ is a group. □

We have already remarked that $\det \mathcal{O} = \pm 1$, $\forall \mathcal{O} \in O(2)$, if we pick the component such that $\det \mathcal{O} = +1$ it is straightforward to show that this subset is also a group; this subgroup is known as the *special orthogonal group* $SO(2) = \{\mathcal{O} \in O(2) \mid \det \mathcal{O} = +1\}$. As a result of this observation we have the simple

Corollary 2.13. *The orthogonal group $O(2)$ may be expressed as the disjoint union of $SO(2)$ and $\{\mathcal{O} \in O(2) \mid \det \mathcal{O} = -1\}$, i.e.,*

$$O(2) = SO(2) \cup \{\mathcal{O} \in O(2) \mid \det \mathcal{O} = -1\}\,.$$

Hence, $SO(2)$ is the identity connected component of $O(2)$.

A typical matrix $SO(2)$ will have the form

$$\mathcal{O}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

where θ is the parameter of the rotation about the origin; clearly, the inverse of each such $\mathcal{O}(\theta)$ by reversing the rotation; i.e. $\mathcal{O}(\theta)^{-1} = \mathcal{O}(-\theta)$. This matrix clearly generates the entire group.

Lemma 2.14. *Any orthogonal matrix $\mathcal{O}(\theta) \in SO(2)$ can be written as the exponential of a single antisymmetric matrix τ;*

$$\mathcal{O}(\theta) = e^{i\theta\tau} \quad \textit{where} \quad \tau = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

We first note that the lemma makes sense on simple dimensional grounds since $e^{i\theta\tau} = \sum_{n=0}^{\infty} \frac{1}{n!}(i\theta\tau)^n$.

Proof of lemma 2.14. We begin by noting that $\tau^2 = \mathbb{1}$ and expanding the exponential

$$e^{\theta\tau} = \left(\sum_{\text{odd}} + \sum_{\text{even}}\right)\frac{1}{n!}(i\theta\tau)^n = \cos\theta\mathbb{1} + i\tau\sin\theta \tag{2}$$

which we see is the general form $\mathcal{O}(\theta)$. □

By varying θ we can continuously map any $\mathcal{O} \in SO(2)$ into any other $\widetilde{\mathcal{O}} \in SO(2)$; hence, the symmetry is said to be continuous.

To recover the other, $\det \mathcal{O} = -1$, piece of $O(2)$ we make the following deduction that any element of $\{\mathcal{O} \in O(2) \mid \det \mathcal{O} = -1\}$ can be written as the product of an $SO(2)$ matrix and

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The matrix P is a mirror symmetry, it maps $(x_1, x_2) \mapsto (x_1, -x_2)$, and is an example of a discrete symmetry. (The symmetry represented by P is the mirror symmetry *parity.*) Hence, we see that the elements of $O(2)$ are all parametrised by a single parameter θ. We say that $O(2)$ is a one parameter group, that is, it is one dimensional.

Elegant as lemma 2.14 is it has not really told us *why* the exponential map is needed here nor has it told us anything about *why* τ takes the form it does. To explain this we need to introduce the notion of infinitesimal generators and some Lie theory.

2.4 Lie groups and infinitesimal generators

Let $\mathcal{G}_0$ be the identity connected component of a Lie group $\mathcal{G}$; we wish to consider group elements which are 'near e'. As all the Lie groups what are of interest to us are matrix Lie groups we content ourselves with the following construction: As $\mathbb{1} \in \mathcal{G}_0$ and $\mathcal{G}$ is a $n \times n$ matrix Lie group, i.e. a group for which we have a notion of 'nearness', we may consider those group elements which are infinitesimaly close to the identity: in particular we consider those elements $g(\delta\theta)$ of the form

$$g(\delta\theta) - \mathbb{1} = O(\delta\theta) ,$$

such a construction is possible because of the continuous nature of the symmetry groups. Such elements $g(\delta\theta)$, for infinitesimal $\delta\theta^i$, are of the form

$$\mathbb{1} + i \sum_{i=1}^{n} \tau^i \delta\theta^i$$

where the set $\{\tau^i\}_i$ are constant coefficient matrices known as the *infinitesimal generators* of the (matrix) Lie group. It follows from Taylor's theorem that τ^i are given by

$$\tau^i = -i[\partial_{\theta^i} g](0) .$$

Remark: We remark that strictly speaking our discussion only applies to the *identity connected* component of the Lie group. However, this (obvious) technical point will not bother us further.

Example 2.15 ($SO(2)$)**.** *The infinitesimal generator of $SO(2)$ is easily seen to be given by*

$$\tau = -i\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array}\right)\Bigg|_{\theta=0} = \left(\begin{array}{cc} 0 & i \\ -i & 0 \end{array}\right)$$

which we saw earlier.

As infinitesimal generators physically correspond to infinitesimal changes in the group one might expect that by performing a large number of them one can recover a finite group element. Indeed, one has the following

Lemma 2.16. *Let $\mathcal{G}_0$ be the identity connected component of a Lie group $\mathcal{G}$ have the following family of infinitesimal generators $\{\tau^i\}_{i\in\iota}$. The generator $g_i(\theta)$ corresponding to each $\{\tau^i\}_{i\in\iota}$ may be recovered by the exponential map, i.e. $g_i(\theta) = \exp(i\theta\tau^i)$.*

Proof. Consider $g(\delta\theta) = \mathbb{1} + i\delta\theta\tau^i$, i.e. an infinitesimal variation about the idenitity; then as a finite variation $g_i(\theta)$ is the result of many applications of inifinitesimal ones we have that

$$g_i(\theta) = \lim_{n\to\infty}\big(g(\delta\theta)\big)^n = \lim_{n\to\infty}\big(\mathbb{1} + i\delta\theta\tau^i\big)^n = \lim_{n\to\infty}\big(\mathbb{1} + i\frac{\theta}{n}\tau^i\big)^n = \exp(i\theta\tau^i)\,.$$

Hence, if we exponentiate the infinitesimal generators we recover the generators and, in this sense, $\mathcal{G}_0$. ☐

We close this section with the useful *Baker-Campbell-Hausdorff formula* which may be stated thus: let A, B be two $n \times n$ matrices then

$$\exp A \exp B = \exp\big(A + B + \frac{1}{2}[A, B] + \ldots\big)\,,$$

a corollary of which is that the Lie group arising as the exponential of an Abelian Lie algebra is also Abelian.

2.5 The rotation group...revisited

There is nothing special about the choice of 2×2 matrices in our discussion, our argument is valid for any $n \in \mathbb{N}$; viz.

Definition. *The set* $\{\mathcal{O} \in \mathrm{GL}_n(\mathbb{R}) \mid \mathcal{O}^{\mathrm{t}}\mathcal{O} = \mathbb{1}\}$ *equipped with matrix multiplication is called the* orthogonal group $O(n)$. *The* special orthogonal group $SO(n) = O(n) \cap \mathrm{SL}_n(\mathbb{R})$.

As $SO(2)$ consisted of continuous rotations which kept $x_1^2 + x_2^2$ invariant it is easy to see that $SO(3)$ consists of continuous rotations which keep $x_1^2+x_2^2+x_3^2$ fixed. The group $SO(3)$ is three dimensional; this follows from the consideration that any $\mathcal{O} \in SO(3)$ has nine elements, three of which are fixed by the condition $\mathcal{O}^{\mathrm{t}} = \mathcal{O}^{-1}$ and the remaining three from the consideration of the determinant. It is straightforward to write down the rotation matrices R_i

$$\begin{aligned}
R_1(\theta) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \\
R_2(\phi) &= \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix}, \\
R_3(\psi) &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

from which we define, in analogy with $SO(2)$, the infinitesimal generators of the rotation group

$$\tau^i = \frac{1}{i}\frac{\mathrm{d}R_i(\theta)}{\mathrm{d}\theta}\bigg|_{\theta=0}.$$

The appearance of the $-i$ is merely convention. The matrices τ^i are given by

$$\tau^1 = -i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \tau^2 = -i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau^3 = -i\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is straightforward to show that these matrices satisfy the commutation relation $[\tau^i, \tau^j] = i\epsilon^{ijk}\tau^k$.

Infinitesimal rotations about some angle $\delta\theta$ about the i axis are given by

$$R_i(\delta\theta) = \mathbb{1} + i\tau^i\delta\theta\,.$$

If we now consider a rotation about the i axis through some angle $\theta = N\delta\theta$ $(N \to \infty)$ we see that

$$R_i(\theta) = (\mathbb{1} + i\tau^i\delta\theta)^N = \big(\mathbb{1} + i\tau^i \frac{\theta}{N}\big)^N \to \exp i\tau^i\theta\,.$$

It is straightforward, but tedious, to confirm that this reproduces the three matrices given earlier.

In order to see how $SO(3)$ acts on functions f we need a representation which respects the group properties and can meanifully act on a function space. This is clearly provided by taking $L_x = yp_z - zp_y$, etc., where $\mathbf{p} = -i\hbar\vec{\nabla}$ is the momentum operator. Then $\exp i\vec{\theta}\cdot\mathbf{L}$ is a representation of $SO(3)$ and we see that *angular momentum is the generator of rotation.*

Example 2.17 (Rotation invariance)**.** *Let $f \in C^1(\mathbb{R}^3)$ be defined by $f(x,y,z) = g(x^2 + y^2 + z^2)$ where $g \in C^1(\mathbb{R})$, then under the left action of* $\exp(i\vec{\theta}\cdot\mathbf{L})$*, $\mathbf{L}$ the angular momentum operator from above and $\vec{\theta} \in \mathbb{R}^3$, $\vec{\theta}\cdot\mathbf{L}f \equiv 0$; hence, $f \mapsto f' = f$ under $SO(3)$.*

2.6 Lie groups and Lie algebras

We have already seen that if we exponentiate the generators of the rotation group we recover the group elements. This is illustrative of a more general piece of theory which we mention now.

Corresponding to every Lie group $\mathcal{G}$ is a *Lie algebra* $\mathfrak{g}$ which is a flat vector space with a *Lie bracket* or commutator defined for a set of vector fields $\{\tau^i\}_{i\in\iota}$ which can serve as the basis for the space.

Definition. *A* Lie algebra *is a vector space V over a field $\mathcal{F}$ equipped with a binary operation $[\cdot,\cdot] : V \times V \mapsto V$ called the* Lie bracket *which satisfies:*
i) bi-linearity $[ax + by, z] = a[x,z] + b[y,z]$ for all scalars $a, b \in \mathcal{F}$ and $x, y, z \in V$;
ii) skew-symmetry, $[x,y] = -[y,x]$ for all $x, y \in V$;
iii) the Jacobi identity *$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x, y, z \in V$.*

We write the Lie bracket as $[\tau^i, \tau^j] = f^{ij}{}_k\tau^k$ where the $f^{ij}{}_k$ are known as the *structure constants* of the Lie algebra. The structure constants uniquely determine the Lie algebra (this powerful result is known as *Lie's theorem*). If the structure constants all vanish then the Lie algebra is said to be *Abelian.* We remark that in the instance that the vector space V has an associative product $*$ defined on it we can identify the Lie bracket with the commutator $[x, y] = x * y - y * x$ for all $x, y \in V$.

Proposition 2.18. *The commutator satisfies the Jacobi identity.*

Proof. The follows from $[x, [y, z]] = x[y, z] - [y, z]x = xyz - xzy - yzx + zyx$ and its cyclic permutations. $\square$

Example 2.19 ($\mathrm{M}_n(\mathcal{F})$)**.** *The linear space* $\mathrm{M}_n(\mathcal{F})$ *of all* $n \times n$ *matrices over the field* $\mathcal{F}$ *equipped with the commutator* $[A, B] = AB - BA$ *where the product is understood to be matrix multiplication is a Lie algebra.*

Example 2.20 ($(\mathbb{R}^3, \times)$)**.** *The cross product* $\times : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ *is an antisymmetric binary mapping which satisfies the Jacobi identity; hence,* $[x, y] = x \times y$ *defines a Lie bracket on* $\mathbb{R}^3$.

Proposition 2.21. *Let* $\{\tau^i\}_i$ *be the set of infinitesimal generators of a matrix Lie group. The matrices* $\{\tau^i\}_i$ *form a Lie algebra.*

Proof. The proof is trivial: we note that $\{\tau^i\}_i$ are square matrices so we *a priori* have the necessary linear space structure, an associative multiplication and commutator. $\square$

It is perhaps useful to remark on a piece of the history of this subject, Lie originally referred to these algebras as *infinitesimal groups*. However insightful this language is to us we continue to adopt the modern parlance.

We denote the Lie algebra produced in the above fashion from the (matrix) Lie group $\mathcal{G}$ by $\mathfrak{g}$. The astute reader will be wondering why we did not simply take the matrix Lie group itself to be its Lie algebra. The reason for this resides within the general theory of Lie algebras and is beyond the scope of this course. The interested reader is directed to §5 of Nakahara M., *Geometry, topology and physics*, Institute of Physics publishing or the previously mentioned Les Houches essay.

Example 2.22 ($\mathfrak{so}(3)$)**.** *The Lie algebra* $\mathfrak{so}(3)$ *corresponding to* $SO(3)$ *is defined by the relation* $[\tau^i, \tau^j] = i\epsilon^{ijk}\tau^k$.

2.7 The special unitary group

In this section we consider another important example in physics.

Definition. *The set* $\{\mathcal{U} \in \mathrm{GL}_n(\mathbb{C}) \mid \mathcal{U}^\dagger \mathcal{U} = \mathbb{1}\}$ *equipped with matrix multiplication forms a group known as the* unitary group $U(n)$. *The subgroup* $SU(n) = U(n) \cap \mathrm{SL}_n(\mathbb{C})$ *is known as the* special unitary group.

We first remark that if we restrict the elements of a $U(n)$ matrix to be real then $U(n) = O(n)$.

We now specialise to $SU(2)$ where we have the following observation:

Proposition 2.23. *There are* 3 *real degrees of freedom associated with a* $SU(2)$ *matrix.*

Proof. Let $a, b, c, d \in \mathbb{C}$ then an element of $\mathrm{GL}_2(\mathbb{C})$ written

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has 8 real degrees of freedom. The requirement that $M^\dagger = M^{-1}$ and $\det M = 1$ is essentially the relations $a = \bar{d}$, $b = -\bar{c}$ and $ad - bc = 1$ which reduce the total degree of freedom by two, two and one respectively. $\square$

Hence, we expect to be able to find 3 independent elements which characterise $SU(2)$.

Example 2.24 ($SU(2)$ and $\mathfrak{su}(2)$)**.** *The set*

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

obey the following commutation relation

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{jik}\frac{\sigma_k}{2}\,.$$

This relation can also be used to define a group. Indeed, using the exponential map we *construct the three group elements*

$$
\begin{aligned}
\mathcal{U}_1(\theta) &= \exp(i\theta\sigma_x/2) = \begin{pmatrix} \cos\theta/2 & i\sin\theta/2 \\ i\sin\theta/2 & \cos\theta/2 \end{pmatrix}, \\
\mathcal{U}_2(\phi) &= \begin{pmatrix} \cos\phi/2 & \sin\phi/2 \\ -\sin\phi/2 & \cos\phi/2 \end{pmatrix}, \\
\mathcal{U}_3(\psi) &= \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}.
\end{aligned}
$$

As such we have family of 2×2 *matrices for whom* $\mathcal{U}_i^\dagger = \mathcal{U}_i^{-1}$*; such matrices are known as* unitary matrices*. Moreover, since* $\det \mathcal{U}_i = +1$*, these matrices are special. In fact, we have constructed the group* $SU(2)$*; the group of* 2×2 *unitary matrices with determinant* $+1$*. This group is known as the* special unitary group*. The Pauli matrices define the Lie algebra* $\mathfrak{su}(2)$*.*

2.8 Recovering the Lie group from a Lie algebra

We have indentified a curious phenomena; namely, that a Lie algebra does not have to correspond to a unique Lie group (c.f. $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$). If we think about the construction of a Lie algebra $\mathfrak{g}$ from $\mathcal{G}_0$ it is not perhaps suprising since the construction is local, i.e., it is based on elements near $e = \mathbb{1}$, and as such is not sensitive to the global characteristics of the group. Hence, that $SO(3)$ and $SU(2)$ have the same Lie algebra is just the statement that locally they are the same. However, they are clearly very different groups.

We state, but do not supply the proof to, the following

Theorem 2.25. *To every Lie algebra there corresponds a unique simply connected Lie group.*

The reader interested in this result, in relation to $SO(3)$ and $SU(2)$, is directed to §3.16 of Schutz B., *Geometrical methods of mathematical physics*, Cambridge.

2.9 Spinors and rotations

We have already seen that $SO(3)$ and $SU(2)$ appear to be related at least locally, i.e.

$$SO(3) \ni \exp(i\theta \cdot \tau) \quad \leftrightarrow \quad \exp(i\theta \cdot \sigma/2) \in SU(2)\,.$$

Therefore, we are motivated to understand the relationship between rotations and $SU(2)$. We shall establish the following:

Proposition 2.26. *Let $\mathbf{x} \in \mathbb{R}^3$ and $M = \mathbf{x} \cdot \sigma$, then a $SU(2)$ transformation $M \mapsto M' = \mathcal{U} M \mathcal{U}^\dagger$ is the same as $\mathbf{x} \mapsto \mathbf{x}' = \mathcal{O}\mathbf{x}$ where $\mathcal{U} \in SU(2)$ and $\mathcal{O} \in SO(3)$.*

Proof. Let $\mathbf{x} \in \mathbb{R}^3$ then we may construct a matrix M from $\mathbf{x}$ using

$$M = \mathbf{x} \cdot \sigma = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}.$$

If we act on M with a $SU(2)$ transform,

$$M \mapsto M' = \mathcal{U} M \mathcal{U}^\dagger\,,$$

then it is clear that $\det M = \det M'$. Moreover, the matrix M' may also be written in the form $\sigma \cdot \mathbf{x}'$. Hence, since $\det M = -|\mathbf{x}|^2$ we conclude that $M \mapsto \mathcal{U} M \mathcal{U}^\dagger$ is the same as a $SO(3)$ transformation on $\mathbf{x}$. □

This is not suprising since we know that the Lie algebra of $SU(2)$ and $SO(3)$ obey the same Lie bracket relation. In fact, by exponentiating the two Lie bracket relations to obtain the group elements

$$SO(3) : \exp(i\theta^i \tau^i) \quad \text{and} \quad SU(2) : \exp(i\theta^i \sigma^i/2)$$

we see, by compairing these two relations, the fundamental distinction between vectors and spinors: *a vector is invariant under a rotation of 2π whereas a spinor needs a 4π rotation to return it to its original configuration.*

Formally, we say, since there are two elements in $SU(2)$ corresponding to a single element in $SO(3)$, that $SU(2)$ is a *double cover* of $SO(3)$.