

## VII.7

### $SO(10)$ Unification

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#### Each family into a single representation

At the end of chapter VII.5 we felt we had good reason to think that  $SU(5)$  unification is not the end of the story. Let us ask if we might be able to fit the 5 and  $10^*$  into a single representation of a bigger group  $G$  containing  $SU(5)$ .

It turns out that there is a natural embedding of  $SU(5)$  into the orthogonal  $SO(10)$  that works,<sup>1</sup> but to explain that I have to teach you some group theory. The starting point is perhaps somewhat surprising: We go back to chapter II.3, where we learned that the Lorentz group  $SO(3, 1)$ , or its Euclidean cousin  $SO(4)$ , has spinor representations. We will now generalize the concept of spinors to  $d$ -dimensional Euclidean space. I will work out the details for  $d$  even and leave the odd dimensions as an exercise for you. You might also want to review appendix B now.

#### Clifford algebra and spinor representations

Start with an assertion. For any integer  $n$  we claim that we can find  $2n$  hermitean matrices  $\gamma_i$  ( $i = 1, 2, \dots, 2n$ ) that satisfy the Clifford algebra

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \tag{1}$$

In other words, to prove our claim we have to produce  $2n$  hermitean matrices  $\gamma_i$  that anticommute with each other and square to the identity matrix. We will refer to the  $\gamma_i$ 's as the  $\gamma$  matrices for  $SO(2n)$ .

For  $n = 1$ , it is a breeze:  $\gamma_1 = \tau_1$  and  $\gamma_2 = \tau_2$ . There you are.

<sup>1</sup> Howard Georgi told me that he actually found  $SO(10)$  before  $SU(5)$ .

Now iterate. Given the  $2n$   $\gamma$  matrices for  $SO(2n)$  we construct the  $(2n + 2)$   $\gamma$  matrices for  $SO(2n + 2)$  as follows

$$\gamma_j^{(n+1)} = \gamma_j^{(n)} \otimes \tau_3 = \begin{pmatrix} \gamma_j^{(n)} & 0 \\ 0 & -\gamma_j^{(n)} \end{pmatrix}, \quad j = 1, 2, \dots, 2n \quad (2)$$

$$\gamma_{2n+1}^{(n+1)} = 1 \otimes \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3)$$

$$\gamma_{2n+2}^{(n+1)} = 1 \otimes \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (4)$$

(Throughout this book 1 denotes a unit matrix of the appropriate size.) The superscript in parentheses is obviously for us to keep track of which set of  $\gamma$  matrices we are talking about. Verify that if the  $\gamma^{(n)}$ 's satisfy the Clifford algebra, the  $\gamma^{(n+1)}$ 's do as well. For example,

$$\begin{aligned} \{\gamma_j^{(n+1)}, \gamma_{2n+1}^{(n+1)}\} &= (\gamma_j^{(n)} \otimes \tau_3) \cdot (1 \otimes \tau_1) + (1 \otimes \tau_1) \cdot (\gamma_j^{(n)} \otimes \tau_3) \\ &= \gamma_j^{(n)} \otimes \{\tau_3, \tau_1\} = 0 \end{aligned}$$

This iterative construction yields for  $SO(2n)$  the  $\gamma$  matrices

$$\gamma_{2k-1} = 1 \otimes 1 \otimes \dots \otimes 1 \otimes \tau_1 \otimes \tau_3 \otimes \tau_3 \otimes \dots \otimes \tau_3 \quad (5)$$

and

$$\gamma_{2k} = 1 \otimes 1 \otimes \dots \otimes 1 \otimes \tau_2 \otimes \tau_3 \otimes \tau_3 \otimes \dots \otimes \tau_3 \quad (6)$$

with 1 appearing  $k - 1$  times and  $\tau_3$  appearing  $n - k$  times. The  $\gamma$ 's are evidently  $2^n$  by  $2^n$  matrices. When and if you feel confused at any point in this discussion you should work things out explicitly for  $SO(4)$ ,  $SO(6)$ , and so on.

In analogy with the Lorentz group, we define  $2n(2n - 1)/2 = n(2n - 1)$  hermitean matrices

$$\sigma_{ij} \equiv \frac{i}{2} [\gamma_i, \gamma_j] \quad (7)$$

Note that  $\sigma_{ij}$  is equal to  $i\gamma_i\gamma_j$  for  $i \neq j$  and vanishes for  $i = j$ . The commutation of the  $\sigma$ 's with each other is thus easy to work out. For example,

$$[\sigma_{12}, \sigma_{23}] = -[\gamma_1\gamma_2, \gamma_2\gamma_3] = -\gamma_1\gamma_2\gamma_2\gamma_3 + \gamma_2\gamma_3\gamma_1\gamma_2 = -[\gamma_1, \gamma_3] = 2i\sigma_{13}$$

Roughly speaking, the  $\gamma_2$ 's in  $\sigma_{12}$  and  $\sigma_{23}$  knock each other out. Thus, you see that the  $\frac{1}{2}\sigma_{ij}$ 's satisfy the same commutation relations as the generators  $J^{ij}$ 's of  $SO(2n)$  (as given in appendix B). The  $\frac{1}{2}\sigma_{ij}$ 's represent the  $J^{ij}$ 's.

As  $2^n$  by  $2^n$  matrices, the  $\sigma$ 's act on an object  $\psi$  with  $2^n$  components that we will call the spinor  $\psi$ . Consider the unitary transformation  $\psi \rightarrow e^{i\omega_{ij}\sigma_{ij}}\psi$  with  $\omega_{ij} = -\omega_{ji}$  a set of real numbers. Then

$$\psi^\dagger \gamma_k \psi \rightarrow \psi^\dagger e^{-i\omega_{ij}\sigma_{ij}} \gamma_k e^{i\omega_{ij}\sigma_{ij}} \psi = \psi^\dagger \gamma_k \psi - i\omega_{ij} \psi^\dagger [\sigma_{ij}, \gamma_k] \psi + \dots$$

for  $\omega_{ij}$  infinitesimal. Using the Clifford algebra we easily evaluate the commutator as  $[\sigma_{ij}, \gamma_k] = -2i(\delta_{ik}\gamma_j - \delta_{jk}\gamma_i)$ . (If  $k$  is not equal to either  $i$  or  $j$  then  $\gamma_k$  clearly commutes